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Absolute instability in viscoelastic mixing layers

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The spatiotemporal linear stability of viscoelastic planar mixing layers is investigated. A one-parameter family of velocity profiles is used as the base state with the parameter, S, controlling the amount of shear and backflow. The influence of viscoelasticity in dilute polymer solutions is modeled with the Oldroyd-B and FENE-P constitutive equations. Both models require the specification of the ratio of the polymer-relaxation and convective time scales (the Weissenberg number, We) and the ratio of solvent and solution viscosities (β). The maximum polymer extensibility, L, must also be specified for the FENE-P model. We examine how the variation of these parameters along with the Reynolds number, Re, affects the minimum value of S at which the flow becomes locally absolutely unstable. With the Oldroyd-B model, the influence of viscoelasticity is shown to be almost fully captured by the elasticity, $E^* \equiv \frac{(1-\beta)We}{D_n}$, and S_{crit} decreases as elasticity is increased, i.e., elasticity is destabilizing. A simple approximate dispersion relation obtained via long-wave asymptotic analysis is shown to accurately capture this destabilizing influence. Results obtained with the FENE-P model exhibit a rich variety of behavior. At large values of the extensibility, L, results are similar to those for the Oldroyd-B fluid as expected. However, when the extensibility is reduced to more realistic values ($L \approx 100$), one must consider the scaled shear rate, $\eta_c \equiv \frac{WeS}{2L}$, in addition to the elasticity. When η_c is large, the base-state polymer stress obtained by the FENE-P model is reduced, and there is a corresponding reduction in the overall influence of viscoelasticity on stability. Additionally, elasticity exhibits a stabilizing effect which is driven by the streamwise-normal perturbation polymer stress. As η_c is reduced, the base-state and perturbation normal polymer stresses predicted by the FENE-P model move towards the Oldroyd-B values, and the destabilizing influence of elasticity observed with the Oldroyd-B model is again present. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4851295]

I. INTRODUCTION

The fluid dynamics of dilute polymer solutions has received sustained interest over several decades due in large part to the drag-reducing properties of polymer additives in turbulent flows.¹ Most studies of moderate or high-Reynolds number flows have focused on channel and pipe flows which have attached shear layers. However, in complex geometries, shear layers may separate, and an improved understanding of instability, transition, and turbulence dynamics in spatially developing viscoelastic free shear layers is desirable.

Here, we analyze the linear stability of parallel planar mixing layers. Azaiez and Homsy² (referred to hereafter as AH) investigated the temporal linear stability of a "stationary" mixing layer (where the two streams have equal speeds but flow in opposite directions) using three different constitutive models to represent the influence of viscoelasticity. They focused on large Reynolds (*Re*) and Weissenberg (*We*) numbers where *We* is the ratio of polymer-relaxation and convective timescales. Viscoelasticity was found to be stabilizing, though the strength of the stabilization was

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sensitive to the choice of constitutive model. An important question which follows is, to what degree do $AH's^2$ conclusions apply to spatially developing mixing layers? We address this question by carrying out an absolute/convective linear stability analysis of a one-parameter family of mixing layer velocity profiles. The profile parameter, *S*, controls the amount of shear and backflow in the base state. These profiles were used by Monkewitz and Huerre³ to study convective instability in inviscid mixing layers while Huerre and Monkewitz⁴ used them to illustrate absolute/convective linear stability analysis. The present study can then be considered to be an extension of AH^2 to spatially developing mixing layers or as an extension of the results in Huerre and Monkewitz to viscoelastic mixing layers.

We use local absolute/convective stability analysis⁵ where the base flow is assumed to be locally parallel, and we search for absolute instabilities – perturbations which grow exponentially in time at the point of excitation. The local approach is best-suited for high-*Re*, slowly spreading shear flows. Flow-spreading effects can be fully captured with global methods,^{6,7} though the computational cost is substantially higher. Juniper *et al.*⁸ discuss the complementary nature of local and global methods in their study of the stability of confined Newtonian wakes.

Mixing-layer stability is, of course, closely related to the stability of jets and wakes. The temporal stability of viscoelastic jets was investigated by Rallison and Hinch⁹ and Miller.¹⁰ There is also an extensive body of work on viscoelastic cylinder wakes. Relevant studies include the global stability analysis of Sahin and Owens,¹¹ the Floquet analysis of Richter *et al.*,¹² and the experimental studies of Cadot and Kumar¹³ and Pipe and Monkewitz.¹⁴ These studies generally found viscoelasticity to be stabilizing, though Ref. 14 observed a destabilizing effect when the polymer concentration was sufficiently high.

We note that there have also been a few numerical and experimental studies on the nonlinear dynamics of viscoelastic mixing layers. Simulations^{15–18} were carried out for stationary mixing layers with elasticity number, $E \equiv \frac{We}{Re} \lesssim 1$, and largely found that disturbance amplitudes were either unaffected or moderately reduced by viscoelasticity. Exceptions include Azaiez and Homsy's¹⁵ observation that there is an intensification of vorticity in the braid region of a two-dimensional mixing layer. In addition, Kumar and Homsy¹⁶ showed that at small elasticity, secondary instability in a three-dimensional mixing layer can be enhanced. Experiments on turbulent mixing layers^{19,20} indicate that large-scale structures are "enhanced" by viscoelasticity while smaller scales are attenuated.

We have restricted our focus to the linear problem and evaluate the influence of viscoelasticity on the spatiotemporal stability of mixing layers. The governing equations and the numerical method used to solve them are described in Sec. II. Then, we present numerical results which illustrate the influence of viscoelasticity on the presence and strength of absolute instability. We use two different constitutive models, and comparisons to long-wavelength asymptotic analysis are used to better understand the numerical results.

II. FORMULATION AND METHODOLOGY

The influence of polymers on flow dynamics is modeled by adding a "polymer-stress" term to the Navier-Stokes equations for incompressible flow,

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{1a}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\beta}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1-\beta}{Re} \frac{\partial a_{ji}}{\partial x_j},$$
(1b)

where u_i is velocity, p is pressure, and $\frac{1-\beta}{R_e}a_{ij}$ is the polymer stress. The Reynolds number is $Re = \frac{U_0\delta}{\nu}$ with velocity scale, U_0 , and length scale, δ , defined below. The kinematic viscosity of the solution is ν , while the ratio of the solvent and solution viscosities is $\beta = \frac{\nu_s}{\nu}$. We have assumed that density is uniform and constant. A constitutive equation for the polymer stress is required to close the system of equations. We will use the Oldroyd-B and FENE-P models. With the Oldroyd-B

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model, a_{ij} satisfies

$$\frac{D_u a_{ij}}{Dt} + \frac{1}{We} a_{ij} = \frac{1}{We} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{2}$$

where $We = \frac{\lambda U_0}{\delta}$ is the Weissenberg number, λ is a polymer relaxation timescale, and $\frac{D_u}{Dt}$ is the upper convected time derivative, $\frac{D_u a_{ij}}{Dt} \equiv \frac{\partial a_{ij}}{\partial t} + u_k \frac{\partial a_{ij}}{\partial x_k} - a_{ik} \frac{\partial u_j}{\partial x_k} - a_{kj} \frac{\partial u_i}{\partial x_k}$. The FENE-P model equations are

$$\frac{D_u c_{ij}}{Dt} = -a_{ij},\tag{3a}$$

$$a_{ij} = \frac{1}{We} \left(\frac{L^2 - 3}{L^2 - c_{kk}} c_{ij} - \delta_{ij} \right),$$
 (3b)

where *L* is the maximum polymer extensibility. Note that c_{ij} and *L* are non-dimensionalized with the molecular length scale, $\sqrt{k_B T/H}$, where k_B is the Boltzmann constant, *T* is the fluid temperature, and *H* is a Hookean spring constant. The Oldroyd-B model equations are recovered from Eq. (3) by taking the limit $L \rightarrow \infty$, and we are using a form of the FENE-P model in which $c_{ij} = \delta_{ij}$ at equilibrium.^{21,22}

The Oldroyd-B model can be derived from a kinetic theory in which polymers are represented as beads connected by Hookean springs. While the Oldroyd-B model can qualitatively reproduce important features of viscoelastic shear flows, the underlying assumption of infinitely extensible springs is limiting: the Oldroyd-B model cannot reproduce shear-thinning and can perform poorly in extensional flows. The FENE-P model is based on finitely extensible springs; it does reproduce shear-thinning and generally performs better than the Oldroyd-B model. Further background on the Oldroyd-B and FENE-P models can be found in Bird *et al.*²³

Returning to the development of the governing equations, we decompose the flow into a steady, parallel base state and small-amplitude perturbations, $f = \overline{F}(x_2) + f'(x_i, t)$. Linearizing the governing equations and assuming perturbations take the form $f' = \tilde{f}(x_2)e^{i(\alpha x_1 - \omega t)}$, we obtain

$$\boldsymbol{A}\tilde{f} = \boldsymbol{B}\frac{d\tilde{f}}{dx_2},\tag{4}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \gamma Re - R_3(1-\beta) + \alpha^2 \beta & -(1-\beta)R_2 & Re\frac{dU_1}{dx_2} - (1-\beta)R_4 & i\alpha Re \\ -i\alpha & 0 & 0 & 0 \\ (1-\beta)S_3 & (1-\beta)S_2 - i\alpha\beta & -\gamma Re - \alpha^2\beta + (1-\beta)S_4 & 0 \end{bmatrix}$$
$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-\beta)R_1 + \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -(1-\beta)S_1 & 0 & Re \end{bmatrix}, \ \gamma \equiv i(\alpha \overline{U}_1 - \omega),$$

and $\tilde{f} = \left[\tilde{u}_1 \frac{d\tilde{u}_1}{dx_2} \tilde{u}_2 \tilde{p}\right]^T$. The components of R_m and S_m are used to relate the polymer stress perturbations to \tilde{f}

$$i\alpha\tilde{a}_{11} + \frac{d\tilde{a}_{12}}{dx_2} = R_1 \frac{d^2\tilde{u}_1}{dx_2^2} + R_2 \frac{d\tilde{u}_1}{dx_2} + R_3\tilde{u}_1 + R_4\tilde{u}_2,$$
(5a)

$$i\alpha\tilde{a}_{12} + \frac{d\tilde{a}_{22}}{dx_2} = S_1 \frac{d^2\tilde{u}_1}{dx_2^2} + S_2 \frac{d\tilde{u}_1}{dx_2} + S_3\tilde{u}_1 + S_4\tilde{u}_2.$$
 (5b)

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Expressions for R_m and S_m are given in Appendix A. We only consider two-dimensional instabilities here. This is a common simplification in absolute/convective instability studies of Newtonian flows where Squire's transformation²⁴ can be applied. There is a modified Squire's transformation for the Oldroyd-B model²⁵ which states that for a given three-dimensional perturbation, there is a two-dimensional perturbation with larger growth rate at the same elasticity (though *Re* and *We* may be different). We are not aware of a similar result for the FENE-P model, and consideration of three-dimensional modes in future studies with this model may be worthwhile.

The base-state velocity profiles are given by

$$\overline{U}_1 = 1 + S \tanh(x_2/2). \tag{6}$$

With these profiles, the velocity scale used to non-dimensionalize the equations, U_0 , is the average of the two free streams, and the length scale, δ , is the momentum thickness. The velocities of the two streams are 1 + S and 1 - S. With S < 1, we have a coflowing mixing layer, while S > 1 gives counterflow. As noted in the Introduction, these profiles were used in earlier linear stability studies of inviscid mixing layers,^{3,4} and they allow us to easily vary the amount of co- or counter-flow. In our calculations, we follow AH² (who used $\overline{U}_1 = \tanh(x_2)$) in varying the elasticity while holding the velocity profile fixed. The base-state polymer stress components for an Oldroyd-B fluid are

$$\overline{A}_{11} = 2We\left(\frac{d\overline{U}_1}{dx_2}\right)^2, \ \overline{A}_{12} = \frac{d\overline{U}_1}{dx_2}, \ \overline{A}_{22} = 0.$$
(7)

Corresponding results for the FENE-P model are more complicated and are given in Appendix A.

The system of equations (4) defines an eigenvalue problem where both α and ω are assumed to be complex, and which we solved using a shooting method with re-orthonormalization.²⁶ The equations were numerically integrated from $x_2 = \pm L_d$ to $x_2 = 0$. We used $L_d = 12$ and verified that our solutions were not affected by increasing L_d . At $x_2 = \pm L_d$, we imposed exponentially decaying free-stream solutions which assume an uniform base flow. These solutions take the form $C_1 e^{\pm \zeta_1 x_2} + C_2 e^{\pm \zeta_2 x_2}$ where $\zeta_1^2 = \alpha^2$, $\zeta_2^2 = \alpha^2 + \frac{\gamma Re}{\beta + \frac{(1-\beta)}{We\gamma + 1}}$, and we require the real part of ζ to be positive, $\Re(\zeta) > 0$. The solutions were matched at $x_2 = 0$ by imposing continuity of \tilde{u}_1 , \tilde{u}_2 , $\frac{d\tilde{u}_1}{dx_2}$, and \tilde{p} . We are principally interested in finding saddle points (which satisfy $\frac{d\omega}{d\alpha} = 0$) that correspond to the pinching of upstream- and downstream-propagating spatial branches. The longtime asymptotic behavior of the impulse response of the mixing layer is dictated by the complex wavenumber and frequency (α_0 and ω_0 , respectively) at these saddle points. Specifically, if the imaginary part of ω_0 is positive, the flow is absolutely unstable.^{4,5} The approach used here to find saddle points is very similar to that used by Monkewitz²⁷ for wake instabilities governed by the Orr-Sommerfeld equation. For each of two (complex) frequencies, ω_1 and ω_2 , two eigenvalues, α^+ and α^{-} , are found which correspond to downstream- and upstream-propagating waves. The four eigenvalues are used to find a nonlinear least-squares fit to a Taylor expansion around the saddle point, $\alpha^{\pm} - \alpha_0 = \pm a_s(\omega - \omega_0)^{\frac{1}{2}} + a_l(\omega - \omega_0)$ where a_s and a_l are fit parameters. Then, two new frequencies are chosen closer to the estimated ω_0 , and the procedure is repeated until the relative change in ω_0 is less than 10⁻⁶. Our codes were validated by reproducing: (i) absolute/convective instability results for inviscid mixing layers from Huerre and Monkewitz;⁴ (ii) absolute/convective results for viscous Newtonian wakes from Monkewitz;²⁷ (iii) temporal instability results for the stationary viscoelastic (Oldroyd-B) mixing layer from AH;² and (iv) temporal neutral-stability curves for viscoelastic (FENE-P) channel flow from Zhang et al.²⁸ Results for the inviscid mixing layer obtained for validation were used as initial guesses for viscoelastic cases with small We, large *Re*, and β close to one.

III. RESULTS

We will first investigate the influence of viscoelasticity on absolute instability using the Oldroyd-B model. Its analytical tractability will be useful for understanding the major qualitative trends 014103-5 P. K. Ray and T. A. Zaki



FIG. 1. Influence of elasticity on S_{crit} , (a) Re = 100, three (indistinguishable) curves for $\beta = 0.5, 0.7, 0.9$ are shown; (b) $\beta = 0.7, Re = 50, 100, 400, 1000$.

observed in the numerical results. We will then see how imposing a maximum extensibility on polymer molecules via the FENE-P model modifies the Oldroyd-B results.

A. Oldroyd-B model

In their analysis of the inviscid mixing layer, Huerre and Monkewitz⁴ found that absolute instability was present when the shear parameter, *S*, exceeded a critical value, $S_{crit} = 1.315$. Figure 1 shows how S_{crit} changes as the elasticity, $E^* \equiv \frac{(1-\beta)We}{Re}$, is increased. In Figure 1(a), the Reynolds number is held fixed at Re = 100 while the viscosity ratio, β , is varied. In Figure 1(b), it is *Re* which varies while the viscosity ratio is set to $\beta = 0.7$. There are several important points to note. First, we see that absolute instability is "enhanced" by elasticity. Absolute instability is found at smaller values of *S* (corresponding to less backflow) as E^* increases. Even a single stream mixing layer (S = 1) may be absolutely unstable if the elasticity is sufficiently large ($E^* \gtrsim 6.8$). This destabilizing effect is one of the main results of this paper and will be discussed further below. In Figure 1(a), we see that using E^* as a measure of viscoelasticity fully accounts for the effect of the viscosity ratio, while Figure 1(b) shows that the Reynolds number has little influence for $Re \ge 50$. AH² showed that the linear stability equations used here can be reduced to an "elastic Rayleigh equation" in the limit of large Reynolds and Weissenberg numbers with $We/Re \sim 1$. The only flow parameter in this equation is the elasticity, E^* . Here, we also see that E^* is the dominant flow parameter.

A more detailed view of the influence of viscoelasticity is obtained by examining the complex wavenumber and frequency (Figure 2). We are primarily interested in the temporal growth rate, ω_i , since its sign determines if absolute instability is possible. The temporal growth rate depends on both the elasticity, E^* , and the profile parameter, S. At large S (S = 2.5, 4), elasticity is predominantly stabilizing, however, there is a change in behavior at smaller values of S: elasticity is destabilizing for S = 1, 1.5 and moderate values of E^* (Figure 2(b)). Further growth-rate curves illustrating this destabilizing effect are shown in Figure 3.

1. Long-wavelength asymptotic analysis

We can gain a better understanding of these qualitative trends by comparing our calculations to results from long-wave (small $|\alpha|$) asymptotic analysis. Our approach closely follows the analysis presented in AH² (which, in turn, closely follows Drazin and Howard²⁹). The main steps are provided here, and the reader is referred to those papers for additional details. Assuming that $We \gtrsim \frac{1}{|\alpha|^2}$ and neglecting terms of order $\frac{\beta}{Re}$ and $\frac{(1-\beta)}{Re}$ at all orders of the analysis, the governing equations can be



FIG. 2. (a)–(d) Influence of elasticity and S on absolute instability, Re = 100; $\beta = 0.7$; —— (thick), S = 1; – –, 1.5; – · –, 2.5; ——, 4.0.



FIG. 3. Influence of elasticity and S on temporal growth rate, Re = 100; $\beta = 0.7$; S = 0.9, 1.0, 1.1, 1.3.

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simplified to

$$\frac{d}{dx_2} \left[(\tilde{U}^2 - \tilde{\nu}\overline{A}_{11}) \frac{d\Phi}{dx_2} \right] - \alpha^2 (\tilde{U}^2 - \tilde{\nu}\overline{A}_{11}) \Phi = 0, \tag{8}$$

where $\tilde{U} = \overline{U}_1 - c$, $c = \frac{\omega}{\alpha}$, $\tilde{\nu} = \frac{1-\beta}{Re}$, $\Phi = \frac{\Psi}{\tilde{U}}$, and Ψ is the perturbation stream function. Note that $\tilde{\nu}\overline{A}_{11} = 2E^* \left(\frac{d\overline{U}_1}{dx_2}\right)^2$, and we assume that E^* is O(1). In neglecting terms of order $\frac{(1-\beta)}{Re}$, we have effectively approximated the perturbation stress as

$$\tilde{a}_{11} \approx 2\overline{A}_{11} \frac{d\Phi}{dx_2} + \frac{d\overline{A}_{11}}{dx_2} \Phi, \\ \tilde{a}_{12} \approx -i\alpha\overline{A}_{11} \Phi, \\ \tilde{a}_{22} \approx 0.$$
(9)

We have carried out numerical stability calculations for S = 1 using this approximation, and the results are very close to the full solution presented in Figure 2 (the maximum difference in ω_i is on the order of 10^{-4}).

Continuing on with the development of the asymptotic equations, for $x_2 > 0$, we define $\chi = e^{\alpha x_2} \Psi$. Similarly, for $x_2 < 0$, we define $\theta = e^{-\alpha x_2} \Psi$. At $x_2 = 0$, the matching condition, $\frac{d\chi}{dx_2}\theta - \chi \frac{d\theta}{dx_2} - 2\alpha\theta\chi = 0$ must be satisfied. Then, expanding χ and θ in powers of α , $\chi = \sum_{n=0}^{\infty} \alpha^n \chi_n$, $\theta = \sum_{n=0}^{\infty} \alpha^n \theta_n$, AH² obtain

$$-\frac{\tilde{U}^{2}(0)}{h(0)} \left[\alpha \left(h^{+} + h^{-} \right) + \alpha^{2} \int_{-\infty}^{+\infty} \frac{\left(h - h^{+} \right) \left(h - h^{-} \right)}{h} dx_{2} - \alpha^{3} \int_{-\infty}^{+\infty} dx_{2} \int_{-\infty}^{x_{2}} \left(h(x_{2}) - h^{+} \right) \left(h\left(\tilde{x}_{2}\right) - h^{-} \right) \left(\frac{1}{h\left(x_{2}\right)} + \frac{1}{h\left(\tilde{x}_{2}\right)} \right) d\tilde{x}_{2} + \dots \right] = 0, \quad (10)$$

with $h = \tilde{U}^2 - \tilde{v}\overline{A}_{11}$, $h^+ = (\overline{U}_1^+ - c)^2$, $h^- = (\overline{U}_1^- - c)^2$, and $\overline{U}_1^{\pm} = \overline{U}_1(x_2 \to \pm \infty)$. AH² simplified the integrals in Eq. (10) by considering the large- E^* limit. Then, $h \approx -2E^* \left(\frac{d\overline{U}_1}{dx_2}\right)^2$ and the dispersion relation at 2nd order is

$$h^{+} + h^{-} = 2E^{*} \int_{-\infty}^{+\infty} \left(\frac{d\overline{U}_{1}}{dx_{2}}\right)^{2} dx_{2}.$$
 (11)

In an appendix in AH,² Hinch² arrived at this result via a different analytical route. The principal difference between the two approaches is that Hinch² takes the large- E^* limit near the beginning of his analysis while AH apply this limit near the end of theirs. Hinch² also noted that Eq. (11) is equivalent to the dispersion relation for a vortex sheet with surface tension if one replaces the right-hand side of Eq. (11) with a non-dimensional surface tension, σ . This important analogy with surface tension helps to provide a physical explanation for the influence of viscoelasticity on mixing layer stability. Now, returning to the task of finding a solution when $E^* \sim O(1)$, the complex phase velocity is expanded as $c = c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots$, and Eq. (10) with the velocity profile from Eq. (6) gives

$$c_0 = 1 + iS, \tag{12a}$$

$$c_1 = -iS\left(\frac{E^*}{3} + 1 - 2I_1\right),\tag{12b}$$

$$c_2 = \frac{i}{4S} \left[2c_1^2 - 4Sc_1(4iI_1 + I_2) - 4S^2I_3 \right].$$
(12c)

The terms I_1 , I_2 , and I_3 , represent integrals which depend only on the elasticity, E^* . Full expressions for these integrals are given in Appendix B, and we evaluate them numerically. The final step is to find saddle points by differentiating Eq. (12) with respect to α and requiring $\frac{\partial \omega}{\partial \alpha} = 0$.

Before looking at the full asymptotic solution (to third order), it is helpful to revisit the large- E^* second order result. In this limit, $I_1 \approx 0$, $c_1 \approx -\frac{iSE^*}{3}$, and there is a saddle point with ω_i $= \frac{3(S^2-1)}{4E^*S}$. This approximation is compared with the numerical linear stability result for S = 1.5 in



FIG. 4. Comparison of stability calculations and asymptotic analysis, Re = 100, $\beta = 0.7$; (a) S = 1.5; —, full numerical calculation; –, 2nd order, large- E^* approximation; –, full 3rd-order approximation; · · , numerical calculation with advection approximation; (b) —, numerical calculation; – · –, full 3rd-order approximation.

Figure 4(a). We see that the approximation approaches the numerical result as E^* increases for $E^* \gtrsim 10$. However, the approximation does not show the correct behavior for smaller values of the elasticity. The principal effect of the large- E^* approximation is to neglect the advection term in the x_2 -momentum equation (the polymer shear-stress term is much larger). We have carried out numerical calculations where the full stability equations have been modified, and this advection term has been removed. The result is included in Figure 4(a). Now, the behavior of the calculation is similar to the large- E^* asymptotic approximation, and it is clear that the balance between the advection of cross-stream momentum and polymer shear-stress is very important. Rallison and Hinch⁹ emphasize this balance in their analysis of temporal sinuous instability in viscoelastic jets. In the mixing layer, the advection/polymer-stress balance helps explain why elasticity becomes stabilizing as S is increased (Figure 2(b)). The polymer stress term in the elastic Rayleigh equation (8) at the mixing-layer centerline is $\frac{E^*S^2}{2}$ while the advection term, \tilde{U}^2 , is relatively insensitive to S. Consequently, at sufficiently large values of S, the polymer stress term becomes dominant and the influence of elasticity becomes purely stabilizing (as it is in the large- E^* limit when S > 1).

The general 3rd-order asymptotic result which uses the full expressions for c_1 and c_2 retains the needed advection term and provides a much better approximation to the numerical solution (Figures 4(a) and 4(b)). Important qualitative details such as the destabilizing effect of elasticity for small and moderate values of E^* are captured. The error in the approximation does increase as the elasticity decreases towards zero, however this is unsurprising as there is a corresponding increase in $|\alpha|$ (Figures 2(c) and 2(d)). We can also extract S_{crit} from the asymptotic result (Figure 5). There is an "offset" in the asymptotic curve which could be reduced by including higher-order terms. However, this is unnecessary as the offset is largely independent of the elasticity. Indeed, scaling the asymptotic result so that it matches the numerical result for an inviscid mixing layer (S_{crit} = 1.315) when $E^* = 0$, we see that the approximation accurately reproduces the influence of elasticity on S_{crit} .

B. FENE-P model

We have now investigated the stability of Oldroyd-B mixing layers in some detail. However, as noted in Sec. II, the Oldroyd-B model has tangible deficiencies. Thus, it is important to obtain results with a more realistic model to test the relevance of the trends observed above. Here, we present results using the FENE-P model which requires the specification of an additional parameter, the maximum polymer extensibility, *L*. For large values of *L*, results should be close to those obtained with the Oldroyd-B model. As *L* is reduced, the (nonlinear) stiffness of the polymer molecules is increased, and one expects a reduction in the influence of elasticity. We will consider a large range of values of *L*, but we are particularly interested in results for $L \sim 100$. Simulations^{30,31} of channel



FIG. 5. Asymptotic approximation for S_{crit} ; ——, numerical calculation; – –, 3rd-order approximation; – · –, scaled 3rd-order approximation.

flows with high- and maximum drag reduction have used values of L between 60 and 120 (see also the discussion on "bead-spring parameters" in Zhou and Akhavan³²).

Figure 6(a) shows how S_{crit} is affected by L with Re = 100 and $\beta = 0.7$. At $L = 10^4$, the Oldroyd-B behavior is recovered. As L is reduced from 10^4 to 3×10^3 , there is a small stabilizing effect. However, the S_{crit} curve rises significantly as L is decreased further to 10^3 and a similar trend is seen in the growth rate for the single-stream mixing layer (Figure 6(b)). Indeed, as can be inferred from Figure 6(a), the single stream mixing layer will not become absolutely unstable at any value of E^* when L is reduced to 10^3 (at these values of Re and β). Figure 7 shows the effect of reducing L further. Reducing L from 1000 to 50 (Figure 7(a)), we see further stabilization. Also notable is the observation that S_{crit} now tends to increase with E^* indicating that elasticity is stabilizing. Reducing L from 50 to 10 (Figure 7(b)) largely removes the influence of elasticity.

For the Oldroyd-B fluid, the influence of Re and β was largely subsumed into the elasticity, E^* . The FENE-P model produces more complicated behavior as can be seen in Figure 8 where β is varied (and L = Re = 100). There is a clear β -dependence in the results which was not present



FIG. 6. Influence of maximum polymer extensibility, L; Re = 100, $\beta = 0.7$, L = 1000, 3000, 5000, 10^4 ; ——(thick), Oldroyd-B; (a) S_{crit} , (b) S = 1.



FIG. 7. Influence of maximum polymer extensibility, L; Re = 100, $\beta = 0.7$; (a) · · ·, L = 1000; —(thick), L = 500; --, L = 200; —, L = 100; - · -, L = 50; (b) L = 10, 20, 50.

in the Oldroyd-B fluid (Figure 1). In addition, the FENE-P results shown thus far indicate that a realistic maximum extensibility ($L \sim 100$) can reduce the influence of viscoelasticity and removes the destabilizing influence seen in the Oldroyd-B results. We can gain a better understanding of these results and construct criteria for the recovery of "Oldroyd-B behavior" (with $L \sim 100$) by examining the base-state and perturbation polymer stresses.

1. Base-state polymer stress

The behavior of the base-state polymer stress in the FENE-P mixing layer is closely related to standard viscometric results for the FENE-P model in steady homogeneous shear flow.³³ These viscometric results show how material functions²³ such as the viscosity and normal stress coefficients depend on the non-dimensional shear rate, $\lambda \dot{\gamma}$, where $\dot{\gamma}$ is the velocity gradient. At low shear rates, the elongation of the FENE springs is modest, and the FENE-P and Oldroyd-B models produce similar results. At higher shear rates, the springs become more stretched, nonlinear elasticity becomes important, and the FENE-P polymer stress components become attenuated relative to their Oldroyd-B counterparts. These trends are present in the tanh mixing layer as well, though the velocity gradient is not constant, and the polymer stress depends on the local shear rate. The role of the extensibility must also be considered, and using Eqs. (7), (A1), and (3b), the ratio of the (non-zero) Oldroyd-B and FENE-P stress components can be shown to be functions of the scaled shear rate,



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FIG. 9. Base-state polymer stress; (a) ratio of values of $\overline{A}_{11}(x_2 = 0)$ obtained with FENE-P and Oldroyd-B models; (b) \overline{A}_{11} and \overline{A}_{12} at $x_2 = 0$ with S = 1 and L = 100; —, FENE-P; – –: Oldroyd-B.

$$\eta \equiv \frac{We\frac{d\overline{U}_1}{dx_2}}{L}$$

$$\overline{4}_{11}^{F} = \frac{1}{[F(\eta)]^2},$$
(13a)

$$\frac{\overline{A}_{12}^F}{\overline{A}_{12}^O} = \frac{1}{F(\eta)},$$
(13b)

where the superscripts "F" and "O" refer to the FENE-P and Oldroyd-B models, respectively. Both models give $\overline{A}_{22} = 0$, and F is defined in the Appendix A 1. At the mixing-layer centerline, $\eta = \eta_c = \frac{WeS}{2L}$, and the function, $\frac{1}{[F(\eta_c)]^2}$, is shown in Figure 9(a). As expected, when the shear rate, η_c , is small, there is little difference between the FENE-P and Oldroyd-B models, and at larger values of η_c , the Oldroyd-B stress is larger. This behavior is also seen in Figure 9(b) where we have shown the dependence of the stress components on We for S = 1, L = 100, and $x_2 = 0$. We can anticipate that the relative attenuation of A_{11} as η is increased is connected to the reduction in the influence of elasticity which was observed in our stability results when L was reduced. Continuing with this line of reasoning, we would expect to recover Oldroyd-B behavior if η_c is less than some critical value, η_c^* . Then, taking $S \approx 1.2$, $L \approx 100$, and $E^* \approx 2$ (where, in Figure 1, we observed the strongest destabilization with the Oldroyd-B model), we obtain the criteria, $Re \lesssim 80(1-\beta)\eta_{h}^{*}$. This indicates that we should look at smaller Reynolds numbers; however, with the parallel flow approximation, the Reynolds number cannot be reduced to arbitrarily small values. Therefore, we consider Re = 10 and first verify that Oldroyd-B trends are not significantly changed by reducing the Reynolds number from 100 to 10. Figure 10(a) shows that the shape of the S_{crit} curve is not significantly changed. There is an upward shift indicating stabilization due to the increased viscous stress, and a small β -dependence appears when $E^* \lesssim 1$. However, our focus is on the destabilizing influence of elasticity which is largely unaffected. Moving on to FENE-P results with L = 100 (Figure 10(b)), we now see a very large β -dependence over a broad range of elasticities. Significantly, for $\beta \lesssim 0.7$ there is now a reduction in S_{crit} as elasticity is increased. The range of E^* over which this behavior is observed depends strongly on β , and for $\beta = 0.9$, elasticity has little influence. When $E^* = 1$, β = 0.9, and Re = 10, we find that $\eta_c = 0.7$ which corresponds to an approximately 50% reduction in \overline{A}_{11} relative to the Oldroyd-B value (Figure 9(a)).

2. FENE-Rayleigh equation

A more concrete view of the dynamics can be obtained by developing approximations similar to those used in the derivation of the elastic Rayleigh equation (8). We assume that both We and L



FIG. 10. Influence of viscosity ratio, β , on S_{crit} ; \cdots , $\beta = 0.3$; --, $\beta = 0.5$; --, $\beta = 0.7$; -, -, $\beta = 0.9$ (a) Oldroyd-B, (b) Re=10; -- (thick), Oldroyd-B; all other curves: FENE-P with L = 100.

are $O(|\alpha|^{-2})$ which leads to the following leading-order approximation for the perturbation stress:

$$a_{11} \approx \left(\frac{2\eta^2}{F} + F\right) \left[\frac{2}{F} \left(\overline{A}_{11}\right) \frac{d\Phi}{dx_2} + \frac{1}{We} \frac{d\overline{C}_{11}}{dx_2} \Phi\right], \quad a_{12} \approx -i\alpha \left(\overline{A}_{11}\right) \Phi, \quad a_{22} \approx 0.$$
(14)

Using this approximation and neglecting the Newtonian viscous stress, the perturbation equations can be simplified to a "FENE-Rayleigh" equation

$$\frac{d}{dx_2} \left[\left(\tilde{U}^2 - \tilde{v}\overline{A}_{11} \left(1 + 4\frac{\eta^2}{F^2} \right) \right) \frac{d\Phi}{dx_2} \right] - \alpha^2 \left(\tilde{U}^2 - \tilde{v}\overline{A}_{11} \right) \Phi = 0.$$
(15)

Before we analyze (15), we should assess how well it and (14) represent the full equations. Numerical solutions of the FENE-Rayleigh equation and of the full equations modified with (14) are compared to solutions of the full equations with L = 100 in Figure 11. At Re = 100, the FENE-Rayleigh equation provides an excellent approximation to the full equations. When Re = 10, the approximation errors are larger. Reducing the Reynolds number increases the importance of the Newtonian viscous stress which was neglected in deriving (15). Similarly, reducing Re while holding E^* constant results in a reduction of We which was assumed to be large when deriving (14). Nevertheless, the maximum error in the FENE-Rayleigh result is less than 10%, this error decreases as E^* increases, and the basic qualitative behavior is well-captured for $E^* \gtrsim 1$.

Equation (15) is very similar to the elastic Rayleigh equation (8), though there are a few important differences. As discussed earlier, the normal stress, \overline{A}_{11} , in the two equations will differ



FIG. 11. Approximations to FENE-P model, $\beta = 0.7$, L = 100, (a) Re = 100, (b) Re = 10; —, full equations; –, equations with approximate perturbation polymer stress; – –, FENE-Rayleigh equation; · · · , modified FENE-Rayleigh equation.



FIG. 12. Influence of Reynolds number on S_{crit} for $\beta = 0.7$, L = 100; —, Re = 100; –, Re = 50; – –, Re = 20; · · · , Re = 10; (b) only:— (thick), Oldroyd-B. The (*)'s represent points where (a) $E^* = 0.5$; (b) We/L = 0.5. The solid, thickened portions of the Re = 10 and Re = 20 curves indicate regions where "Oldroyd-B behavior" is expected.

according to (13a). There is also a new term in the FENE-Rayleigh equation, $\tilde{\nu}\overline{A}_{11}(4\frac{\eta^2}{F^2})\frac{d\Phi}{dx_2}$, which is associated with the streamwise-normal perturbation polymer-stress, \tilde{a}_{11} (cf. Eq. (14)). This term represents the generation of \tilde{a}_{11} at moderate-to-large shear rates due to the action of the velocity perturbations on the background normal stress. The importance of this effect can be assessed by comparing full FENE-Rayleigh calculations to calculations with the term removed. Results from such a comparison are included in Figure 11. At both Reynolds numbers, we see that the " \tilde{a}_{11} term" has a strong stabilizing effect. At Re = 100, it is almost entirely responsible for the (stabilizing) influence of viscoelasticity. The \tilde{a}_{11} term becomes important when $4\frac{\eta^2}{F^2} \gtrsim 1$. At moderate and large shear rates, $F(\eta) \approx 2^{1/3} \eta^{2/3}$ is a very good approximation (cf. the large shear-rate approximations for material functions in steady shear flow in Bird *et al.*³³). It follows that $4\frac{\eta^2}{F^2} \approx (4\eta)^{2/3}$, and the \tilde{a}_{11} term makes an increasingly large contribution to the overall influence of viscoelasticity as the shear rate increases.

We can also use the FENE-Rayleigh equation to gain an improved understanding of when Oldroyd-B behavior is recovered. The FENE-Rayleigh equation simplifies to the elastic Rayleigh equation if $F \approx 1$ and $\eta \ll \frac{1}{2}$. These conditions are not independent and, if one is satisfied, so is the other. In the discussion of the base-state stress above, we argued that Oldroyd-B behavior would be seen if $\eta_c < \eta_c^*$, and here, we have now obtained a similar, better-defined constraint for η . The centerline shear rate, $\eta_c = \frac{WeS}{2L}$, contains, S, and since S_{crit} is an unknown, it is preferable to work with a constraint which only contains independent variables. If we take $S \approx 1.2$ as a "typical" value, the requirement, $\frac{We}{L} \ll \frac{1}{1.2}$ is obtained. However, it is helpful to relax this criteria and to look for "Oldroyd-B-like" behavior if $\frac{We}{L} \lesssim \frac{1}{2}$. Satisfying this constraint does not by itself lead to elasticity exhibiting a destabilizing influence. In Figure 10(a), we see that the elasticity must also be sufficiently large, and we set $E^* \gtrsim 0.5$ as a second requirement. To test the utility of these two constraints, we have carried out calculations at $Re = 10, 20, 50, \text{ and } 100 \text{ with } \beta = 0.7$ (Figure 12). When Re = 50 and $E^* \ge 0.5$, $\frac{We}{L} \ge \frac{5}{6}$ and the constraints cannot both be satisfied. Thus, elasticity should not be destabilizing, and this is consistent with the numerical result. The same argument applies at Re = 100. When Re = 20 and $E^* \ge 0.5$, $\frac{We}{L} \ge \frac{1}{3}$, so now, destabilization is expected when $\frac{1}{3} \leq \frac{We}{L} \leq \frac{1}{2}$. This portion of the numerical solution is "highlighted" in Figure 12(a), and a destabilizing influence is indeed observed. At the lowest Reynolds number, Re = 10, the constraints indicate that the "region of destabilization" is larger which is confirmed by the numerical results. In Figure 12(b), we also see that in this (highlighted) region the behavior of the Re = 10 FENE-P solution is very similar to that of the Oldroyd-B solution. The simple constraints developed here provide a helpful framework for understanding earlier results as well. In Figure 10(b), when $\beta = 0.9$, we do not see any destabilization. For this case, We/L < 0.5 corresponds to $E^* < 0.5$, and the criteria $E^* > 0.5$ cannot also be satisfied. In Figure 7(a), our criteria require

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 $0.5 \leq E^* \leq 1.5$ for L = 1000, and this roughly corresponds to the region where viscoelasticity has a substantive destabilizing influence. Larger values of elasticity correspond to larger shear rates, and for $E^* \gtrsim 4$, elasticity exhibits a stabilizing influence. At these larger shear rates, \overline{A}_{11} and \tilde{a}_{11} are modified by finite-extensibility and nonlinear elasticity (as discussed above) resulting in this stabilization. Analogous behavior is seen in the L = 1000 growth-rate curve in Figure 6(b).

IV. CONCLUDING REMARKS

The influence of viscoelasticity on absolute instability in locally parallel planar mixing layers has been investigated. We focused on the effect of the elasticity, E^* , on the critical value of the base velocity profile parameter, S_{crit} , at which absolute instability first appears, and results with both the Oldroyd-B and FENE-P constitutive models were presented. With the Oldroyd-B model, we found that elasticity has a destabilizing influence in that S_{crit} decreases as elasticity is increased. This result from a spatiotemporal stability analysis of spatially developing mixing layers should be contrasted with previous temporal-stability studies of Oldroyd-B mixing layers and jets where elasticity was found to have only a stabilizing influence.^{2,9} Comparisons between numerical results and asymptotic approximations provided a fuller understanding of the observed trends. For large values of the elasticity, our results approached the large- E^* , long-wave asymptotic approximation derived by AH.² For moderate values of E^* , full 3rd-order asymptotic results which account for the advection of cross-stream momentum were needed to capture the influence of elasticity.

Results obtained with the FENE-P model exhibited more complex behavior. At moderate Reynolds numbers ($Re \sim 100$), results are similar to the Oldroyd-B model for sufficiently large L ($L \gtrsim 3000$). However, reducing L to more realistic values ($L \sim 100$) resulted in significant changes as the overall influence of viscoelasticity was reduced and increasing elasticity resulted in an increase in S_{crit} – elasticity had become stabilizing. These trends were explained by examining how the scaled shear rate, $\eta = \frac{We^{\frac{dU_1}{dL}}}{L}$, affected the streamwise-normal polymer stress. As η is increased, the base-state normal polymer stress is attenuated (relative to the Oldroyd-B stress), and the influence of elasticity is reduced. Furthermore, the streamwise-normal perturbation polymer stress becomes more important and has a stabilizing influence. We also showed that reducing the Reynolds number leads to the appearance of regions in the parameter space where the shear rate is sufficiently small and the elasticity is sufficiently large for the destabilizing influence of elasticity observed in the Oldroyd-B results to be recovered when L = 100.

This study provides an improved understanding of the dynamics of mixing layers of dilute polymer solutions. A number of simplifying assumptions were made, and the relaxation of these assumptions provides a path for further progress. Understanding the importance of base-flow spreading, confinement, and nonlinearity are all important as is the consideration of closely related flows such as wakes and jets. Finally, we note that shear flows of polymer melts are of substantial industrial importance. These fluids typically have very large viscosities resulting in flows with near-zero Reynolds numbers. Consequently, the analysis presented here must be modified appropriately for low-Re, rapidly spreading shear layers.

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APPENDIX A: FURTHER DETAILS ON FORMULATION AND METHODOLOGY

1. Base state polymer stress for FENE-P model

Expressions for the base state conformation tensor are taken from Sureshkumar et al..³⁴

$$\overline{C}_{11} = \frac{1}{F} \left(1 + \frac{2\tau^2}{F^2} \right),\tag{A1a}$$

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(A2)

$$\overline{C}_{12} = \frac{\tau}{F^2},\tag{A1b}$$

$$\overline{C}_{22} = \overline{C}_{33} = \frac{1}{F},\tag{A1c}$$

where $F = \frac{3\Omega}{2sinh(\phi/3)}$, $\phi = sinh^{-1}(\frac{3}{2}\sqrt{3}\Omega)$, $\Omega = \sqrt{2}\frac{\tau}{L}$, and $\tau = We\frac{d\overline{U}_1}{dx_2}$. Substituting \overline{C}_{ij} into Eq. (3b) gives \overline{A}_{ij} .

2. Expressions for perturbation polymer stress

In Sec. II, we presented the governing equations in terms of R_m and S_m (Eqs. (4) and (5)). Here, we give expressions for these quantities for the Oldroyd-B model and describe how they are computed for the FENE-P model. Note that quantities with subscripts *m* and *n* are not physical vectors or tensors though summation of repeated indices is implied.

a. Oldroyd-B model

The linearized Oldroyd-B model gives the following relation between $\tilde{a}_m = [\tilde{a}_{11} \ \tilde{a}_{12} \ \tilde{a}_{22}]^T$ and $\tilde{g}_m = \left[\frac{d\tilde{u}_1}{dx_2} \ \tilde{u}_1 \ \tilde{u}_2\right]^T$:

$$A_{mn}\tilde{a}_n=B_{mn}\tilde{g}_n,$$

$$A_{mn} = \begin{bmatrix} \psi & -2\tau & 0 \\ 0 & \psi & -\tau \\ 0 & 0 & \psi \end{bmatrix}, B_{mn} = \begin{bmatrix} 2\tau & 2i\alpha(\overline{A}_{11}We+1) & -We\frac{d\overline{A}_{11}}{dx_2} \\ 1 & 0 & -We\frac{d^2\overline{U}_1}{dx_2^2} + i\alpha(\overline{A}_{11}We+1) \\ 0 & -2i\alpha & 2i\alpha\tau \end{bmatrix},$$

where $\psi = 1 + We\gamma$, and $\tau = We\frac{d\overline{U}_1}{dx_2}$. Inverting A_{mn} , we have $\tilde{a}_m = M_{mn}\tilde{g}_n$, $M_{mn} = (A_{mp})^{-1}B_{pn}$. It follows that, $\frac{d\tilde{a}_m}{dx_2} = \frac{dM_{mn}}{dx_2}\tilde{g}_n + M_{mn}\frac{d\tilde{g}_n}{dx_2}$. Comparing these expressions with Eq. (5) and using continuity, we obtain

$$R_{m} = \begin{bmatrix} M_{21} \\ i\alpha M_{11} + N_{21} + M_{22} \\ i\alpha (M_{12} - M_{23}) + N_{22} \\ i\alpha M_{13} + N_{23} \end{bmatrix}, S_{m} = \begin{bmatrix} M_{31} \\ i\alpha M_{21} + N_{31} + M_{32} \\ i\alpha (M_{22} - M_{33}) + N_{32} \\ i\alpha M_{23} + N_{33} \end{bmatrix}, N_{mn} = \frac{dM_{mn}}{dx_{2}}.$$
 (A3)

b. FENE-P model

The development of the linearized FENE-P equations is similar to but more complicated than the development of the Oldroyd-B equations above. The perturbation conformation tensor components, $\tilde{c}_m = [\tilde{c}_{11} \ \tilde{c}_{12} \ \tilde{c}_{22} \ \tilde{c}_{33}]^T$, are related to the velocity, $\tilde{g}_m = \left[\frac{d\tilde{u}_1}{dx_2} \ \tilde{u}_1 \ \tilde{u}_2\right]^T$, by $A_{mn}\tilde{c}_n = WeB_{mn}\tilde{g}_n$,

$$A_{mn} = \begin{bmatrix} We\gamma + H_0(1 + \overline{H}_{11}) & -2\tau & H_0\overline{H}_{11} & \overline{H}_{11} \\ H_0\overline{H}_{12} & We\gamma + H_0 & -\tau + H_0\overline{H}_{12} & \overline{H}_{12} \\ H_0\overline{H}_{22} & 0 & We\gamma + H_0(1 + \overline{H}_{22}) & \overline{H}_{22} \\ H_0\overline{H}_{33} & 0 & H_0\overline{H}_{33} & We\gamma + H_0(1 + \overline{H}_{33}) \end{bmatrix},$$
(A4)

$$B_{mn} = \begin{bmatrix} 2\overline{C}_{12} & 2i\alpha\overline{C}_{11} & -\frac{dC_{11}}{dx_2} \\ \overline{C}_{22} & 0 & -\frac{d\overline{C}_{12}}{dx_2} + i\alpha\overline{C}_{11} \\ 0 & -2i\alpha\overline{C}_{22} & 2i\alpha\overline{C}_{12} - \frac{d\overline{C}_{22}}{dx_2} \\ 0 & 0 & -\frac{d\overline{C}_{33}}{dx_2} \end{bmatrix},$$

where $\tau = We \frac{d\overline{U}_1}{dx_2}$, $H_0 = \frac{L^2 - 3}{L^2 - \overline{C}_{kk}}$, and $\overline{H}_{ij} = \frac{\overline{C}_{ij}}{L^2 - \overline{C}_{kk}}$. As with the Oldroyd-B model, we invert A_{mn} to get $\tilde{c}_m = D_{mn}\tilde{g}_n$, $D_{mn} = We(A_{mp})^{-1}B_{pn}$. We then have, $\frac{d\tilde{c}_m}{dx_2} = \frac{dD_{mn}}{dx_2}\tilde{g}_n + D_{mn}\frac{d\tilde{g}_n}{dx_2}$. The perturbation stress, $\tilde{a}_m = [\tilde{a}_{11} \tilde{a}_{12} \tilde{a}_{22}]^T$ is related to \tilde{c}_n by

$$\tilde{a}_{m} = \frac{H_{0}}{We} E_{mn} \tilde{c}_{n},$$

$$E_{mn} = \begin{bmatrix} 1 + \overline{H}_{11} & 0 & \overline{H}_{11} & \overline{H}_{11} \\ \overline{H}_{12} & 1 & \overline{H}_{12} & \overline{H}_{12} \\ \overline{H}_{22} & 0 & 1 + \overline{H}_{22} & \overline{H}_{22} \end{bmatrix},$$
(A5)

so $\tilde{a}_m = M_{mp}\tilde{g}_p$, $M_{mp} = \frac{H_0}{We}E_{mn}D_{np}$, and the desired expressions for R_m , S_m are again given by (A3).

APPENDIX B: EXPRESSIONS FOR INTEGRALS FROM LONG-WAVE ASYMPTOTICS

In Sec. III, results for the complex phase velocity were given in terms of integrals I_1 , I_2 , and I_3 . The full expressions for these terms are

$$I_1 = \int_{-\infty}^{+\infty} \frac{dz}{g},\tag{B1a}$$

$$I_2 = \int_{-\infty}^{+\infty} \left[\tanh(z) + i - \frac{4(\tanh(z) - i)}{g^2} \right] dz, \tag{B1b}$$

$$I_{3} = \int_{-\infty}^{+\infty} \left[(g+2i) \int_{-\infty}^{z} (g(\tilde{z}) - 2i) \left(\frac{1}{g(z)} + \frac{1}{g(\tilde{z})} \right) d\tilde{z} \right] dz,$$
(B1c)

where $g = (\tanh(z) - i)^2 - \frac{E^*}{2} (\tanh^2(z) - 1)^2$.

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