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An exact representation of the nonlinear triad interaction terms in spectral space

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Spectral analysis of the Navier–Stokes equations requires treatment of the convolution of pairs of Fourier transforms $\hat{f}$ and $\hat{g}$. An exact, tractable representation of the nonlinear terms in spectral space is introduced, and relies on the definition and manipulation of a combination matrix. A spectral energy equation is derived where the nonlinear triad interactions are expressed using the combination matrix. The formulation is applied to the problem of homogeneous, isotropic turbulence. By finding the solution in an appropriate canonical basis, the energy spectrum in the inertial range $E(k) \sim \epsilon^{2/3}k^{-5/3}$ is derived from the Navier–Stokes equations without invoking dimensional scaling arguments.

Key words: homogeneous turbulence, isotropic turbulence, turbulence theory

1. Introduction

The complexity of turbulence has inspired novel techniques to examine its dynamics. Three approaches have collectively shaped our understanding of turbulence to date: initially, progress was possible only through inventive theoretical modelling and analysis of carefully measured experimental data. Later, large-scale numerical simulations became feasible and have continued to provide a remarkable level of detail in resolving turbulence. While non-intrusive and fully resolved measurements are the primary hurdle to experiments, numerical simulations are constrained by the ability to access high Reynolds numbers. The two techniques have therefore been complementary and have arguably outpaced progress in theory, which has been hindered by the difficulties associated with the nonlinear terms in the Navier–Stokes equations. In one of the most fundamental flow configurations, namely homogeneous isotropic turbulence, both experiments and numerical simulations have conclusively verified the robustness of the $-5/3$ 'law' for the wavenumber energy spectrum in the inertial range (Sreenivasan 1995; Yeung & Zhou 1997). On the other hand, theoretical deductions from the Navier–Stokes equations have remained elusive. As such, the $-5/3$ 'law' is often referred to as the Kolmogorov phenomenology (Domaradzki & Rogallo 1990; Zhou 1993*a*) because its derivation has remained reliant on scaling assumptions. In this work, the derivation of the $-5/3$ scaling is tackled directly,

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starting from the Navier–Stokes equations, and by invoking the minimum number of assumptions.

The study of decaying homogeneous isotropic turbulence often starts by invoking a Fourier representation of the turbulence fields. In Fourier space, the nonlinear advection term in the Navier–Stokes equations becomes a convolution integral, and a triad interaction term in the turbulence kinetic energy equation. Despite being conservative, and therefore contributing only to energy exchange between Fourier modes, this term is at the heart of many of the interesting questions in the literature. For example, the original phenomenology by Kolmogorov (1941*a*,*b*) suggests a cascading flux of energy from one wavenumber to another. However, it is not clear from the nonlinear terms in the energy equation whether the energy flux and the associated triad interactions are indeed local in wavenumber space. Studies have therefore focused on evaluating the spectral energy transfer from direct numerical simulations (Domaradzki & Rogallo 1990; Zhou 1993*a*,*b*). Similarly, the nonlinear term plays a central role in the sweeping hypothesis (Kraichnan 1964), where large-scale eddies advect smaller eddies past an Eulerian observer, causing 'broadening' of the energy frequency spectra (Tennekes 1975). This view, however, conflicts with the Kolmogorov phenomenology of local interactions in wavenumber space. The problem is further muddied by the finite correlation between the large and small wavenumbers (Praskovsky *et al.* 1993). Notwithstanding these conceptual challenges, the decay of the energy wavenumber spectra in the inertial range has demonstrated a robust $-5/3$ slope (Sreenivasan 1995; Yeung & Zhou 1997), as predicted by the Kolmogorov phenomenology. It is, for instance, independent of the sweeping hypothesis since the action of large-scale eddies on smaller ones does not appear in the energy wavenumber spectra, but rather affects only the higher-order spectra (Zhou, Praskovsky & Vahala 1993; Zhou, Matthaeus & Dmitruk 2004). Despite its robustness, the $-5/3$ 'law' has not been possible to derive directly from the Navier–Stokes equations.

The primary difficulty in working with the spectral Navier–Stokes equations is appropriately accounting for all nonlinear interactions. An analytical treatment requires some means of tracking energy transfer from two arbitrary wavenumbers $\boldsymbol{p}$ and $\boldsymbol{q}$ into a third wavenumber $\boldsymbol{k}$. Previous attempts have relied on either a direct computation of all triad interactions or phenomenological models for the modal energy transfer. However, an exact and tractable analytical representation for triad interactions in spectral space is still missing and is sought herein. Using the proposed formulation, we then derive the spectral energy equation for homogeneous, isotropic turbulence and derive the $-5/3$ decay rate in the inertial range.

A description of the fundamental studies and literature regarding energy transfer in homogeneous isotropic turbulence can be found in the review by Sreenivasan & Antonia (1997) and the text by Lesieur (2008). Of particular relevance is the work discussing triad interactions and the detailed behaviour of the energy transfer function (Orszag 1970; Domaradzki & Rogallo 1990; Waleffe 1992, 1993). These studies raised the importance of considering local versus non-local triad interactions in simulations of isotropic turbulence (Yeung, Brasseur & Wang 1995; Zhou & Rubinstein 1996). Several phenomenological models have also been developed to help understand the nature of these interactions. For instance, Domaradzki (1992) introduced a self-similar scaling for the modal energy transfer, and the eddy-damped quasinormal Markovian (EDQNM) theory of Orszag (1970) has been verified by the results of Ohkitani & Kida (1992). Spectral eddy viscosity models have also been used since the work of Heisenberg (1948). More recent work in large-eddy simulations (Lesieur, Montmory

& Chollet 1987; Métais & Lesieur 1992) has also employed models which account for energy transfer in the unresolved part of the wavenumber spectrum.

The principal goal of this work is to introduce a new methodology for expressing the nonlinear terms in the Navier–Stokes equations and, using this analytical approach, to derive the $-5/3$ law from the Navier–Stokes equations without any scaling arguments and with minimal assumptions. The methodology is introduced in § 2, where the convolution of two Fourier transforms $\hat{f}$ and $\hat{g}$ is expressed using a combination matrix. In § 3, we formulate the Navier–Stokes equations in spectral space with explicit dependence on the wavenumber $\boldsymbol{k}$. In § 4, the corresponding spectral energy equation is developed for homogeneous, isotropic turbulence, and an analytical solution for the energy spectrum in the inertial range is found.

2. The Fourier representation

At the most fundamental level, the challenge of nonlinearity stems from the spectral representation for the product of two functions $h(x) = f(x)g(x)$. Here, the functions $h(x)$, $f(x)$ and $g(x)$ are assumed to be periodic and sufficiently well behaved to permit the following discrete Fourier representation,

$$f(x) = \sum_{p=-\infty}^{\infty} \hat{f}^p \mathrm{e}^{\mathrm{i}px}, \quad g(x) = \sum_{q=-\infty}^{\infty} \hat{g}^q \mathrm{e}^{\mathrm{i}qx}, \quad h(x) = \sum_{k=-\infty}^{\infty} \hat{h}^k \mathrm{e}^{\mathrm{i}kx}. \tag{2.1a–c}$$

In the notation $\hat{a}_i^p$, the superscript p refers to the pth Fourier component, $-\infty \leqslant p \leqslant +\infty$, and the subscript i refers to the ith spatial dimension, $i = \{1, 2, 3\}$. Throughout this analysis, the summation convention is not used in order to avoid confusion.

Inserting the Fourier representation (2.1) into the multiplication $h(x) = f(x)g(x)$, we can determine the composition of each Fourier component $\hat{h}^k$,

$$\hat{h}^k = \sum_{p+q=k} \hat{f}^p \hat{g}^q = \sum_{p=-\infty}^{\infty} \hat{f}^p \hat{g}^{k-p}. \tag{2.2}$$

Here, in the first equality, the sum occurs over each triplet of indices that satisfy the condition $p + q = k$, while in the second equality, the coefficients $\hat{g}$ are offset from $\hat{f}$ by $k - 2p$. The above statement is the discretized convolution $\hat{h} = \hat{f} * \hat{g}$.

The ultimate goal is to determine the relationship between $\hat{h}^k$ and any other coefficient $\hat{h}^{k'}$ without any restrictions or offsets. A simpler representation of the summations (2.2) is therefore needed, and is achieved by introducing the combination matrix

$$C^{pq,k} = \begin{cases} 0, & p + q \neq k \\ 1, & p + q = k \end{cases} \tag{2.3}$$

which merely codifies the convolution operation in matrix form. Note that the combination matrix can be related to the Kronecker delta via $C^{pq,k} = \delta^{p+q,k}$. Using (2.3), the summation (2.2) can be written in the bilinear form

$$\hat{h}^k = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{f}^p C^{pq,k} \hat{g}^q. \tag{2.4}$$

The form in (2.4) involves a summation over all p and q, with all of the indicial unsightliness handled by $C^{pq,k}$. Much further mathematical manipulation is possible if we adopt the infinite dimensional vectors

$$\hat{\boldsymbol{h}} = \begin{pmatrix} \vdots \\ \hat{h}^{-1} \\ \hat{h}^{0} \\ \hat{h}^{+1} \\ \vdots \end{pmatrix}, \quad \hat{\boldsymbol{f}} = \begin{pmatrix} \vdots \\ \hat{f}^{-1} \\ \hat{f}^{0} \\ \hat{f}^{+1} \\ \vdots \end{pmatrix}, \quad \hat{\boldsymbol{g}} = \begin{pmatrix} \vdots \\ \hat{g}^{-1} \\ \hat{g}^{0} \\ \hat{g}^{+1} \\ \vdots \end{pmatrix}, \tag{2.5a–c}$$

and start to examine $C^{pq,k}$ in matrix form. If k is treated as a fixed parameter, then the individual elements of $C^{pq,k}$ can be written in terms of an infinite dimensional matrix, e.g. for $k=0$, $k=-1$, $k=+1$, and so forth:

$$\left[C^{pq,0}\right] = \boldsymbol{C}^{,0} = \begin{pmatrix} & & & \ddots & \ddots \\ & 0 & 0 & 1 & \ddots \\ & 0 & 1 & 0 & \\ \ddots & 1 & 0 & 0 & \\ \ddots & \ddots & & & \end{pmatrix}, \tag{2.6a}$$

$$\boldsymbol{C}^{,-1} = \begin{pmatrix} & & \ddots & \ddots & \ddots \\ & 0 & 1 & 0 & \\ \ddots & 1 & 0 & & \\ \ddots & 0 & & & \\ \ddots & & & & \end{pmatrix}, \quad \boldsymbol{C}^{,+1} = \begin{pmatrix} & & & & \ddots \\ & & & 0 & \ddots \\ & & 0 & 1 & \ddots \\ & 0 & 1 & 0 & \\ \ddots & \ddots & \ddots & & \end{pmatrix}. \tag{2.6b,c}$$

At this point a pattern emerges: the diagonal of 1s in $\boldsymbol{C}^{,0}$ appears directly on the southwest-northeast diagonal, while for $\boldsymbol{C}^{,-1}$ it lies directly above it, and for $\boldsymbol{C}^{,+1}$, all 1s appear directly below it. Furthermore, a demotion operator $\boldsymbol{\mathcal{D}} = [\mathcal{D}^{pq}]$ and a promotion operator $\boldsymbol{\mathcal{P}} = [\mathcal{P}^{pq}]$ are defined in terms of the permutation matrices,

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ \ddots & 1 & 0 & & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{\mathcal{P}} = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & 0 & 1 & 0 & \\ & & 0 & 1 & \ddots \\ & & & 0 & \ddots \\ & & & & \ddots \end{pmatrix}. \tag{2.7a,b}$$

These operators relate $\boldsymbol{C}^{,\mp 1}$ to $\boldsymbol{C}^{,0}$ via $\boldsymbol{C}^{,-1} = \boldsymbol{\mathcal{D}}\boldsymbol{C}^{,0}$ and $\boldsymbol{C}^{,+1} = \boldsymbol{\mathcal{P}}\boldsymbol{C}^{,0}$. Taking this one step further, we can see that $\boldsymbol{C}^{,k+1} = \boldsymbol{\mathcal{P}}\boldsymbol{C}^{,k}$, and that any matrix $C^{pq,k}$ can be related to the original $C^{pq,0}$ after repeated application of the $\boldsymbol{\mathcal{P}}$ operator:

$$\boldsymbol{C}^{,k} = (\boldsymbol{\mathcal{P}})^{k}\, \boldsymbol{C}^{,0}. \tag{2.8}$$

Finally, from the definition of the operators $\mathcal{D}$ and $\mathcal{P}$ in (2.7), it can also be verified that $\mathcal{D}\mathcal{P} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the infinite dimensional identity operator. Therefore, only one of the two operators needs to be defined, since $\mathcal{D} = (\mathcal{P})^{-1}$. Once (2.8) is inserted into (2.4), we can rewrite the expression for $\hat{h}^k$ as

$$\hat{h}^k = \hat{\boldsymbol{f}} \cdot (\mathcal{P})^k \boldsymbol{C}^{\cdot 0} \hat{\boldsymbol{g}}. \tag{2.9}$$

One final step is needed before we can demonstrate the utility of this approach in the context of the Navier–Stokes equations. The matrix power $(\mathcal{P})^k$ in (2.9) is expressed in terms of a diagonal matrix in order to explicitly show the dependence of k inside each $\hat{h}^k$. This uses the fact that $(\mathcal{P})^k = \boldsymbol{\Sigma}(\boldsymbol{\Lambda})^k(\boldsymbol{\Sigma})^{-1}$, where the column vectors of $\boldsymbol{\Sigma}$ are the eigenvectors of $\mathcal{P}$, and $\boldsymbol{\Lambda}$ is a diagonal matrix composed of the eigenvalues λ_i of $\mathcal{P}$. Inserting this expression into (2.9) yields

$$\hat{h}^k = \hat{\boldsymbol{f}} \cdot \boldsymbol{\Sigma}(\boldsymbol{\Lambda})^k \boldsymbol{\Sigma}^{-1} \boldsymbol{C}^{\cdot 0} \hat{\boldsymbol{g}}. \tag{2.10}$$

The powers of the matrix $\boldsymbol{\Lambda}$ can be easily computed. In fact, because the eigenvalues of any permutation matrix have a unit magnitude ($|\lambda_i| = 1$), we can see that

$$(\boldsymbol{\Lambda})^k = \begin{pmatrix} \ddots & & & & \\ & \mathrm{e}^{\mathrm{i}k\theta_{-1}} & & & \\ & & \mathrm{e}^{\mathrm{i}k\theta_0} & & \\ & & & \mathrm{e}^{\mathrm{i}k\theta_1} & \\ & & & & \ddots \end{pmatrix} \tag{2.11}$$

where $0 \leqslant \theta_i < 2\pi$. Hence, it is possible to see from (2.11) that the difference between $\hat{h}^k$ and any other component $\hat{h}^{k'}$ lies only in the phase factors in the matrix $(\boldsymbol{\Lambda})^k$. In the next section, we apply these concepts to the nonlinear terms of the Navier–Stokes equations.

3. The Navier–Stokes equations

In this section, we seek to rewrite the Navier–Stokes equations in spectral space using the formulation introduced in § 2. The governing equations for an incompressible, viscous fluid in physical space are

$$\nabla \cdot \boldsymbol{u} = 0; \quad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{3.1a,b}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity, $p(\boldsymbol{x}, t)$ is the pressure, and we assume constant density ρ and kinematic viscosity ν. For the purposes of this study, we also assume that the flow is homogeneous in all three dimensions within a sufficiently large periodic domain $L \times L \times L$, such that $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ can be written in terms of Fourier expansions

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}^{\boldsymbol{k}}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \quad p(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{p}^{\boldsymbol{k}}(t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{3.2a,b}$$

In three dimensions, $\boldsymbol{k} = (k_1, k_2, k_3)$ and $\hat{\boldsymbol{u}}^{\boldsymbol{k}} = \left(\hat{u}_1^{\boldsymbol{k}}, \hat{u}_2^{\boldsymbol{k}}, \hat{u}_3^{\boldsymbol{k}}\right)$. The magnitude of the wavevector $\boldsymbol{k}$ is given by $k^2 = |\boldsymbol{k}|^2 = k_1^2 + k_2^2 + k_3^2$. Since the velocity field is real valued,

the Fourier coefficients in (3.2) must obey the relation $\hat{\boldsymbol{u}}^{\boldsymbol{k}} = (\hat{\boldsymbol{u}}^{-\boldsymbol{k}})^*$, where * denotes the complex conjugate. To multiply two velocity fields $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$, we also define the combination matrix $C^{\boldsymbol{pq},\boldsymbol{k}}$ analogously to (2.3),

$$C^{\boldsymbol{pq},\boldsymbol{k}} = \begin{cases} 0, & \boldsymbol{p}+\boldsymbol{q} \neq \boldsymbol{k} \\ 1, & \boldsymbol{p}+\boldsymbol{q} = \boldsymbol{k}. \end{cases} \tag{3.3}$$

From (3.3), the following properties of $C^{\boldsymbol{pq},\boldsymbol{k}}$ are evident:

$$C^{\boldsymbol{pq},\boldsymbol{k}} = C^{\boldsymbol{qp},\boldsymbol{k}}, \quad C^{\boldsymbol{pq},\boldsymbol{k}} = C^{(-\boldsymbol{p})(-\boldsymbol{q}),-\boldsymbol{k}}, \quad C^{\boldsymbol{pq},\boldsymbol{k}} = C^{\boldsymbol{p}(-\boldsymbol{k}),-\boldsymbol{q}}, \quad \sum_{\boldsymbol{r}} C^{\boldsymbol{pr},\boldsymbol{k}} C^{\boldsymbol{rq},\boldsymbol{k}} = \delta^{\boldsymbol{pq}}. \tag{3.4a–d}$$

As in the previous section, we can also relate any combination matrix $C^{\boldsymbol{pq},\boldsymbol{k}}$ to the zero-wavenumber $C^{\boldsymbol{pq},\boldsymbol{0}}$ through a set of promotion operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, such that $C^{\boldsymbol{pq},\boldsymbol{k}} = (\mathcal{P}_1)^{k_1}(\mathcal{P}_2)^{k_2}(\mathcal{P}_3)^{k_3} C^{\boldsymbol{pq},\boldsymbol{0}}$. Note that it is not possible to write the operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ in any unique row–column format; they depend on how one orders the (x, y, z) elements in $C^{\boldsymbol{pq},\boldsymbol{0}}$. Regardless, the same properties discussed in § 2 still apply.

The continuity equation (3.1*a*) is simple to express in spectral form, where a differentiation of (3.2*a*) yields $\sum_i k_i \hat{u}_i^{\boldsymbol{k}} = 0$. The pressure equation is also straightforward to derive. By taking the divergence of (3.1*b*) and invoking the continuity constraint, the pressure Poisson equation $\nabla^2 p = -\rho \boldsymbol{\nabla} \cdot (\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})$ is obtained. Inserting (3.2), and using (3.3) in this equation produces the equivalent form in spectral space:

$$\hat{p}^{\boldsymbol{k}} = \frac{\rho}{k^2} \sum_{i,j} \sum_{\boldsymbol{p},\boldsymbol{q}} (p_i q_j) \left(\hat{u}_j^{\boldsymbol{p}} C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_i^{\boldsymbol{q}} \right). \tag{3.5}$$

Lastly, for the nonlinear convective term in (3.1*b*), it can be written in spectral space as

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})^{\boldsymbol{k}} = \mathrm{i} \sum_{\boldsymbol{p},\boldsymbol{q}} \begin{pmatrix} \left(\hat{u}_1^{\boldsymbol{p}} q_1 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_1^{\boldsymbol{q}} + \hat{u}_2^{\boldsymbol{p}} q_2 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_1^{\boldsymbol{q}} + \hat{u}_3^{\boldsymbol{p}} q_3 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_1^{\boldsymbol{q}} \right) \\ \left(\hat{u}_1^{\boldsymbol{p}} q_1 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_2^{\boldsymbol{q}} + \hat{u}_2^{\boldsymbol{p}} q_2 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_2^{\boldsymbol{q}} + \hat{u}_3^{\boldsymbol{p}} q_3 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_2^{\boldsymbol{q}} \right) \\ \left(\hat{u}_1^{\boldsymbol{p}} q_1 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_3^{\boldsymbol{q}} + \hat{u}_2^{\boldsymbol{p}} q_2 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_3^{\boldsymbol{q}} + \hat{u}_3^{\boldsymbol{p}} q_3 C^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_3^{\boldsymbol{q}} \right) \end{pmatrix}. \tag{3.6}$$

However, the expression in (3.6) is still not sufficiently compact to manipulate easily. For instance, the x convective term can be written in a bilinear matrix form as

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})_1^{\boldsymbol{k}} = \mathrm{i} \sum_{\boldsymbol{p},\boldsymbol{q}} \begin{pmatrix} \hat{u}_1^{\boldsymbol{p}} \\ \hat{u}_2^{\boldsymbol{p}} \\ \hat{u}_3^{\boldsymbol{p}} \end{pmatrix} \cdot \begin{pmatrix} q_1 C^{\boldsymbol{pq},\boldsymbol{k}} & 0 & 0 \\ q_2 C^{\boldsymbol{pq},\boldsymbol{k}} & 0 & 0 \\ q_3 C^{\boldsymbol{pq},\boldsymbol{k}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_1^{\boldsymbol{q}} \\ \hat{u}_2^{\boldsymbol{q}} \\ \hat{u}_3^{\boldsymbol{q}} \end{pmatrix}. \tag{3.7}$$

Here, each element appearing in a vector or matrix is assumed to be infinite dimensional, so the matrices in the middle of (3.7) are 3×3 block matrices, where the matrix $q_i C^{\boldsymbol{pq},\boldsymbol{k}}$ is also infinite-dimensional. If we define

$$\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}} = \begin{pmatrix} 0 & 0 & q_1 C^{\boldsymbol{pq},\boldsymbol{k}} \\ 0 & 0 & q_2 C^{\boldsymbol{pq},\boldsymbol{k}} \\ 0 & 0 & q_3 C^{\boldsymbol{pq},\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{I} & 0 \\ 0 & 0 & \boldsymbol{I} \\ \boldsymbol{I} & 0 & 0 \end{pmatrix}^m \tag{3.8}$$

where the powers $m = 1, 2, 3$ and $\boldsymbol{I} \equiv \delta^{\boldsymbol{pq}}$ is the infinite identity matrix, then we can compactly represent the entire nonlinear convective term as

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u})_m^{\boldsymbol{k}} = \mathrm{i} \sum_{\boldsymbol{p},\boldsymbol{q}} \sum_{i,j} \hat{u}_i^{\boldsymbol{p}} \boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}} \hat{u}_j^{\boldsymbol{q}}. \tag{3.9}$$

Inserting (3.5) and (3.9) into (3.1*b*), the final form of the spectral Navier–Stokes is

$$\sum_m k_m \hat{u}_m^{\boldsymbol{k}} = 0, \tag{3.10a}$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_m^{\boldsymbol{k}} + \mathrm{i}\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\hat{u}_i^{\boldsymbol{p}}\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_j^{\boldsymbol{q}} = \frac{-\mathrm{i}k_m}{k^2}\sum_{i,j}\sum_{\boldsymbol{p},\boldsymbol{q}}(p_i q_j)\left(\hat{u}_j^{\boldsymbol{p}} C^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_i^{\boldsymbol{q}}\right). \tag{3.10b}$$

By introducing $C^{\boldsymbol{pq},\boldsymbol{k}}$, we have thus far expressed the Navier–Stokes in a spectral form (3.10) which explicitly shows the dependence on the wavenumber $\boldsymbol{k}$. Note that no approximations have been made other than that the representation (3.2) holds.

4. The spectral energy equation

In this section, we demonstrate the utility of the new representation of the nonlinear terms in spectral space, and derive some insight into the energetics of the flow. We focus on homogeneous, isotropic decaying turbulence and derive analytical solutions to an otherwise impenetrable problem.

The absence of a mean flow leads to one immediate simplification. In the summations over all wavenumbers $\sum_{\boldsymbol{p},\boldsymbol{q}}(\bullet)$, we do not include the zero wavenumbers $\boldsymbol{p} = 0$ and $\boldsymbol{q} = 0$, since the coefficients $\hat{\boldsymbol{u}}^{\boldsymbol{0}}$ are zero in this case. If a mean flow is present in the system, these wavenumbers would be handled separately, or a different basis would be selected.

To form an energy equation, we first define an inner product,

$$(\boldsymbol{a}^{\boldsymbol{k}},\, \boldsymbol{b}^{\boldsymbol{k}}) = \sum_{i=1}^{3} a_i^{\boldsymbol{k}}\left(b_i^{\boldsymbol{k}}\right)^*. \tag{4.1}$$

We take the dot product of $\hat{\boldsymbol{u}}^{\boldsymbol{k}}$ and (3.10*b*), and eliminate the pressure term through the use of the continuity equation,

$$\left(\frac{\partial \hat{u}_m^{\boldsymbol{k}}}{\partial t} + \nu k^2\hat{u}_m^{\boldsymbol{k}} + \mathrm{i}\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\hat{u}_i^{\boldsymbol{p}}\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_j^{\boldsymbol{q}},\, \hat{u}_m^{\boldsymbol{k}}\right) + \text{c.c.} = 0 \tag{4.2}$$

where c.c. stands for the complex conjugate. Using the notation $(\hat{\boldsymbol{u}}^{\boldsymbol{k}},\, \hat{\boldsymbol{u}}^{\boldsymbol{k}}) = |\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2$, equation (4.2) can be written as

$$\frac{\partial}{\partial t}|\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2 + 2\nu k^2|\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2 + \mathrm{i}G^{\boldsymbol{k}} = 0 \tag{4.3}$$

where the nonlinear term $G^{\boldsymbol{k}}$ is defined to be

$$G^{\boldsymbol{k}} = \sum_m\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\hat{u}_i^{\boldsymbol{p}}\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_j^{\boldsymbol{q}}\left(\hat{u}_m^{\boldsymbol{k}}\right)^* - \sum_m\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\left(\hat{u}_i^{\boldsymbol{p}}\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_j^{\boldsymbol{q}}\left(\hat{u}_m^{\boldsymbol{k}}\right)^*\right)^*. \tag{4.4}$$

The nonlinear term $G^{\boldsymbol{k}}$ at first appears unwieldy, but it can be manipulated into a more meaningful form. Using some algebra and the fact that $\hat{\boldsymbol{u}}^{\boldsymbol{k}} = (\hat{\boldsymbol{u}}^{-\boldsymbol{k}})^*$, it is rewritten as

$$G^{\boldsymbol{k}} = \sum_m\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\left(\hat{u}_i^{\boldsymbol{p}}\boldsymbol{\Gamma}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}\hat{u}_j^{\boldsymbol{q}}\left(\hat{u}_m^{\boldsymbol{k}}\right)^*\right) - \sum_m\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{i,j}\left(\left(\hat{u}_i^{-\boldsymbol{p}}\right)\left(\boldsymbol{\Gamma}_{ji,m}^{\boldsymbol{qp},\boldsymbol{k}}\right)^*\left(\hat{u}_j^{-\boldsymbol{q}}\right)\left(\hat{u}_m^{-\boldsymbol{k}}\right)^*\right). \tag{4.5}$$

This allows us to shift the indices by adopting the new wavenumbers $\boldsymbol{p}' = -\boldsymbol{p}$, $\boldsymbol{q}' = -\boldsymbol{q}$, $\boldsymbol{k}' = -\boldsymbol{k}$, and using the identity $\boldsymbol{\Gamma}^{\boldsymbol{qp},\boldsymbol{k}}_{ji,m} = -\boldsymbol{\Gamma}^{\boldsymbol{q'p'},\boldsymbol{k'}}_{ji,m}$. The result is

$$G^{\boldsymbol{k}} = \sum_{m} \sum_{\boldsymbol{p},\boldsymbol{q}} \sum_{i,j} \left(\hat{u}^{\boldsymbol{p}}_i \left(\boldsymbol{\Gamma}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} + \boldsymbol{\Gamma}^{\boldsymbol{qp},\boldsymbol{k}}_{ji,m} \right) \hat{u}^{\boldsymbol{q}}_j \cdot \left(\hat{u}^{\boldsymbol{k}}_m \right)^* \right). \tag{4.6}$$

By introducing the symmetric matrix $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} = \boldsymbol{\Gamma}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} + \boldsymbol{\Gamma}^{\boldsymbol{qp},\boldsymbol{k}}_{ji,m}$, the spectral energy equation (4.2) can be expressed as

$$\left(\frac{\partial}{\partial t} + 2\nu k^2 \right) \hat{\mathcal{E}}(\boldsymbol{k}) = -\mathrm{i} \sum_{m} \sum_{\boldsymbol{p},\boldsymbol{q}} \sum_{i,j} \left(\hat{u}^{\boldsymbol{p}}_i \mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} \hat{u}^{\boldsymbol{q}}_j \right) \hat{u}^{\boldsymbol{k}}_m, \tag{4.7}$$

where $\hat{\mathcal{E}}(\boldsymbol{k}) \equiv |\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2$. This form of the equation is preferable to that given by (4.2). Not only are the nonlinear interactions entirely contained within the symmetric matrix $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$, but also the sum over these nonlinear terms appears exactly in the quadratic form $Q = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}$, which will be exploited in the next section.

4.1. *Reducing the system to canonical form*

In (4.7), the triad interaction term involving $\hat{\boldsymbol{u}}^{\boldsymbol{p}}$, $\hat{\boldsymbol{u}}^{\boldsymbol{q}}$ and $\hat{\boldsymbol{u}}^{\boldsymbol{k}}$ was retained on the right-hand side. The nonlinear term acts to redistribute energy among the wavenumbers $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{k}$, and the utility of the matrices $C^{\boldsymbol{pq},\boldsymbol{k}}$, $\boldsymbol{\Gamma}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ and $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ will be crucial to its subsequent analysis. On the left-hand side of (4.7) are the viscous term and the temporal evolution of $\hat{\mathcal{E}}$.

Rather than solve (4.7) in its current form, we choose to adopt a basis for $\hat{u}^{\boldsymbol{p}}_i$ where the term $\hat{u}^{\boldsymbol{p}}_i \mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} \hat{u}^{\boldsymbol{q}}_j$ appears in canonical form. In doing so, off-diagonal products of $\hat{u}^{\boldsymbol{p}}_i$ and $\hat{u}^{\boldsymbol{q}}_j$ are eliminated and (4.7) simplifies substantially. However, instead of simply diagonalizing the matrix $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ alone, we seek to find a basis where both $\sum \hat{u}^{\boldsymbol{p}}_i \mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} \hat{u}^{\boldsymbol{q}}_j$ and the supplementary quadratic form $\sum \hat{u}^{\boldsymbol{p}}_i \mathcal{W}^{\boldsymbol{pq}}_{ij} \hat{u}^{\boldsymbol{q}}_j$ are reduced to canonical form. From standard linear algebra, finding the basis where both $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}$ and $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x}$ are reduced to canonical form is possible if $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x}$ is positive-definite, and is equivalent to solving the generalized eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \alpha \boldsymbol{W}\boldsymbol{x}$. The desired basis $\boldsymbol{\xi}$ then satisfies the condition $\boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{\xi} = \alpha \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{\xi}$, where α can be determined through the eigenvalues of the matrix $\boldsymbol{W}^{-1}\boldsymbol{A}$. Note that finding such a basis also yields the stationary values of $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{x}$, i.e. points where the quadratic form is locally maximum or minimum.

An appropriate choice for the weight matrix $\mathcal{W}^{\boldsymbol{pq}}_{ij}$ is crucial. Using $\mathcal{W}^{\boldsymbol{pq}}_{ij} = C^{\boldsymbol{pq},\boldsymbol{0}} \delta_{ij}$ results in $\sum \hat{u}^{\boldsymbol{p}}_i \mathcal{W}^{\boldsymbol{pq}}_{ij} \hat{u}^{\boldsymbol{q}}_j = |\hat{\boldsymbol{u}}|^2$ and maximizes the quadratic form with respect to the standard energy norm. However, this leads to some immediate problems because the eigenvalues of the matrix $\left[C^{\boldsymbol{pq},\boldsymbol{0}} \delta_{ij} \right]^{-1} \mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ are not well behaved. In $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ and $\boldsymbol{\Gamma}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$, the entries of the matrices are directly proportional to p_i, meaning that the elements of the matrix approach infinity the further one ventures away from the centre of the matrix sub-block – leading to unbounded eigenvalues. The remedy is to use $\mathcal{W}^{\boldsymbol{pq}}_{ij} = pC^{\boldsymbol{pq},\boldsymbol{0}} \delta_{ij}$. This operator represents the absolute value of flux terms which involve the $\boldsymbol{k} = 0$ wavenumber in the triad interactions. Using this definition of $\mathcal{W}^{\boldsymbol{pq}}_{ij}$, we find the canonical basis with respect to

$$\sum_{\boldsymbol{p},\boldsymbol{q}} \sum_{i,j} \hat{u}^{\boldsymbol{p}}_i \mathcal{W}^{\boldsymbol{pq}}_{ij} \hat{u}^{\boldsymbol{q}}_j = \sum_{\boldsymbol{p}} p |\hat{\boldsymbol{u}}^{\boldsymbol{p}}|^2 = \sum_{\boldsymbol{p}} p \hat{\mathcal{E}}(\boldsymbol{p}). \tag{4.8}$$

Inserting this canonical basis in (4.7) results in the following equation for $\hat{\mathcal{E}}$:

$$\frac{\partial}{\partial t}\hat{\mathcal{E}}(\boldsymbol{k}) + 2\nu k^2 \hat{\mathcal{E}}(\boldsymbol{k}) = -\mathrm{i}\sum_m \left(\sum_{\boldsymbol{p}} \alpha_m^{\boldsymbol{p}} p \hat{\mathcal{E}}(\boldsymbol{p})\right) \hat{u}_m^{\boldsymbol{k}}, \tag{4.9}$$

where $\alpha_m^{\boldsymbol{p}}$ are the eigenvalues of the matrix $\left[\mathcal{W}_{ij}^{\boldsymbol{pq}}\right]^{-1} \mathcal{A}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}} = \left[pC^{\boldsymbol{pq},\boldsymbol{0}}\delta_{ij}\right]^{-1} \mathcal{A}_{ij,m}^{\boldsymbol{pq},\boldsymbol{k}}$. As shown in appendix A, these eigenvalues are $O(1)$, are independent of both m and the wavenumber $\boldsymbol{k}$, and can be used to simplify (4.9) to the following form:

$$\frac{\partial}{\partial t}\hat{\mathcal{E}}(\boldsymbol{k}) + 2\nu k^2 \hat{\mathcal{E}}(\boldsymbol{k}) = -\mathrm{i}\sum_m \overline{\alpha} \left(\sum_{\boldsymbol{p}} p \hat{\mathcal{E}}(\boldsymbol{p})\right) \hat{u}_m^{\boldsymbol{k}}, \tag{4.10}$$

where $\overline{\alpha}$ is the averaged component of $\alpha_m^{\boldsymbol{p}}$ and is independent of $\boldsymbol{p}$.

To convert the summations into an integral over a continuous variable, we split the sum over the three-dimensional index $\boldsymbol{p} = (p_1, p_2, p_3)$ into two separate pieces,

$$\sum_{\boldsymbol{p}} p\hat{\mathcal{E}}(\boldsymbol{p}) = \sum_{p'=0}^{\infty} p' \sum_{|\boldsymbol{p}|=p'} \hat{\mathcal{E}}(\boldsymbol{p}). \tag{4.11}$$

The inner summation occurs over all $\hat{\mathcal{E}}(\boldsymbol{p})$ for which $|\boldsymbol{p}| = p'$, and the outer sum totals all contributions from these spherical shells from $0 \leqslant p' < \infty$. The interior sum can be equivalently stated as the area of the shell $4\pi p'^2$ multiplied by the average energy $\langle\hat{\mathcal{E}}\rangle$ of modes in the shell:

$$\sum_{|\boldsymbol{p}|=p'} \hat{\mathcal{E}}(\boldsymbol{p}) = 4\pi p'^2 \langle\hat{\mathcal{E}}\rangle. \tag{4.12}$$

Here, as in Domaradzki & Rogallo (1990), we define the average of a function $f(\boldsymbol{k})$ as $\langle f(\boldsymbol{k})\rangle = \left(\sum_k f(\boldsymbol{k})\right)/N(k)$, where the number of modes in each shell is given by $N(k) = 4\pi k^2$. The spherically symmetric spectral energy density is defined as

$$E(k) = 4\pi k^2 \left\langle \tfrac{1}{2}|\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2 \right\rangle. \tag{4.13}$$

Combining this average and (4.13) allows us to rewrite the spectral density as

$$E(k) = \frac{4\pi k^2}{N(k)} \sum_{|\boldsymbol{k}|=k} \frac{1}{2}|\hat{\boldsymbol{u}}^{\boldsymbol{k}}|^2 = \frac{1}{2}\sum_{|\boldsymbol{k}|=k} \hat{\mathcal{E}}(\boldsymbol{k}), \tag{4.14}$$

which inserted into (4.11) yields

$$\sum_{\boldsymbol{p}} p\hat{\mathcal{E}}(\boldsymbol{p}) = 2\sum_{p'=0}^{\infty} p'E(p'). \tag{4.15}$$

We now convert the sum over p into a continuous integral and assume the wavenumbers p and k vary continuously from 0 to ∞,

$$\sum_{\boldsymbol{p}} p\hat{\mathcal{E}}(\boldsymbol{p}) \approx 2\int_0^{\infty} p'E(p')\mathrm{d}p' = 2\left(\int_0^{\tilde{k}} p'E(p')\mathrm{d}p' + \int_{\tilde{k}}^{\infty} p'E(p')\mathrm{d}p'\right). \tag{4.16}$$

In (4.16), we have split the integral into two intervals: the first includes the energy containing range and the inertial range of the energy cascade, and the second is over the dissipation range of the spectrum. In this analysis we define a wavenumber $\tilde{k}$ near the boundary between the inertial range and the dissipation range. From previous experiments and computations we know that in the dissipation range ($k > \tilde{k}$), the spectral energy density decays exponentially, while in the inertial range ($k < \tilde{k}$) the decay follows a power law. Hence, we assume the second integral contributes less and can be neglected.

Inserting the results of (4.16) into (4.9) gives

$$\frac{\partial}{\partial t}\hat{\mathcal{E}}(\boldsymbol{k}) = -\mathrm{i}\sum_m \alpha_m \left(2\int_0^{\tilde{k}} p' E(p')\mathrm{d}p'\right)\hat{u}_m^{\boldsymbol{k}} \tag{4.17}$$

after neglecting the contribution of the viscous term in the inertial range. Summing this equation over all wavenumbers $\boldsymbol{k}$, and then multiplying each side by its complex conjugate yields

$$\left(\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\hat{\mathcal{E}}(\boldsymbol{k})\right)\left(\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\hat{\mathcal{E}}(\boldsymbol{k})\right)^* = \left|\left(\sum_{\boldsymbol{k}}\mathrm{i}\sum_m \alpha_m\left(2\int_0^{\tilde{k}} p' E(p')\mathrm{d}p'\right)\hat{u}_m^{\boldsymbol{k}}\right)\right|^2. \tag{4.18}$$

The right-hand side of (4.18) can be related to the energy dissipation rate ϵ by examining the summation over $\boldsymbol{k}$ and using the definition of the spectral energy density $E(k)$:

$$\frac{\partial}{\partial t}\sum_{\boldsymbol{k}}\hat{\mathcal{E}}(\boldsymbol{k}) = \frac{\partial}{\partial t}\sum_k\sum_{|\boldsymbol{k}|=k}\hat{\mathcal{E}}(\boldsymbol{k}) \approx 2\frac{\partial}{\partial t}\int_0^{\tilde{k}} E(k)\mathrm{d}k = 2\epsilon. \tag{4.19}$$

This simplifies (4.18) to

$$\begin{aligned}(2\epsilon)^2 \;=\; &\left(\sum_{\boldsymbol{k}}\mathrm{i}\sum_m \alpha_m\left(2\int_0^{\tilde{k}} p' E(p')\mathrm{d}p'\right)\hat{u}_m^{\boldsymbol{k}}\right)\\ &\times\left(\sum_{\boldsymbol{k}'}-\mathrm{i}\sum_n \alpha_n\left(2\int_0^{\tilde{k}} pE(p)\mathrm{d}p\right)(\hat{u}_n^{\boldsymbol{k}'})^*\right).\end{aligned} \tag{4.20}$$

On the right-hand side of (4.20), a summation over two wavenumbers $\boldsymbol{k}$ and $\boldsymbol{k}'$ occurs:

$$\epsilon^2 = \sum_{\boldsymbol{k},\boldsymbol{k}'}\alpha^2\left(\int_0^{\tilde{k}} pE(p)\mathrm{d}p\right)^2\sum_{m,n}\hat{u}_m^{\boldsymbol{k}}(\hat{u}_n^{\boldsymbol{k}'})^*. \tag{4.21}$$

After examining the sums in terms of spherical shells with $k = |\boldsymbol{k}|$ and $k' = |\boldsymbol{k}'|$, the only terms which survive in the case of isotropic homogeneous turbulence are those for which $\boldsymbol{k} = \boldsymbol{k}'$ (e.g. § 9.2 of Durbin & Reif 2001), leading to

$$\epsilon^2 = \alpha^2\left(\int_0^{\tilde{k}} p' E(p')\mathrm{d}p'\right)^2\sum_k\sum_{|\boldsymbol{k}|=k}\sum_{m,n}\hat{u}_m^{\boldsymbol{k}}\left(\hat{u}_n^{\boldsymbol{k}}\right)^*. \tag{4.22}$$

This can also be written in terms of integrals over the spectral energy density $E(k)$ as

$$\epsilon = \sqrt{2}\alpha \left(\int_0^{\tilde{k}} k' E(k') \mathrm{d}k' \right) \left(\int_0^{\tilde{k}} E(k') \mathrm{d}k' \right)^{1/2}. \tag{4.23}$$

At this point, (4.23) is a nonlinear integral equation for $E(k)$ which can be solved by taking the derivative with respect to $\tilde{k}$:

$$\tilde{k} E(\tilde{k}) \left(\int_0^{\tilde{k}} E(k') \mathrm{d}k' \right)^{1/2} + \frac{E(\tilde{k})}{2} \left(\int_0^{\tilde{k}} k' E(k') \mathrm{d}k' \right) \left(\int_0^{\tilde{k}} E(k') \mathrm{d}k' \right)^{-1/2} = 0. \tag{4.24}$$

Here we have assumed that $\tilde{k}$ is sufficiently large such that ϵ is independent of $\tilde{k}$. Multiplying through by $\left(\int_0^{\tilde{k}} E(k') \mathrm{d}k' \right)^{1/2}$ and simplifying the algebra results in the equation

$$\frac{1}{2} \left(\int_0^{\tilde{k}} k' E(k') \mathrm{d}k' \right) + \left(\int_0^{\tilde{k}} E(k') \mathrm{d}k' \right) \tilde{k} = 0. \tag{4.25}$$

Taking two further derivatives of the equation with respect to $\tilde{k}$ gives this equation in terms of the simple first-order ODE

$$\frac{3}{2} \tilde{k} \frac{\mathrm{d}}{\mathrm{d}\tilde{k}} E(\tilde{k}) + \frac{5}{2} E(\tilde{k}) = 0. \tag{4.26}$$

The solution to (4.26) is $E(\tilde{k}) \sim \tilde{k}^{-5/3}$, which, when introduced into (4.23), produces the expected energy spectrum in the inertial range (Kolmogorov 1941*b*):

$$E(\tilde{k}) = \mathfrak{C} \epsilon^{2/3} \tilde{k}^{-5/3}. \tag{4.27}$$

The present analysis relied on introducing the combination matrix $C^{pq,k}$, and using it to express the nonlinear terms of the Navier–Stokes. Throughout the analysis, the triad-interaction terms are retained in their entirety without any modelling assumptions. By recasting them in canonical form, we can accurately predict the behaviour of the energy spectra in the inertial range for homogeneous isotropic turbulence at high Reynolds number. Only two principal assumptions were introduced, namely that $E(k)$ decays sufficiently fast for $k > \tilde{k}$ and that ϵ is independent of $\tilde{k}$. As such, (4.27) was derived without any scaling arguments, and by appealing directly to the Navier–Stokes equations.

5. Conclusions

In this study, we introduced a new methodology for manipulating the nonlinear terms in the Navier–Stokes equations. Using this methodology, we have determined the appropriate canonical form for expressing the nonlinear terms, and presented an analytical derivation of the $-5/3$ law from the Navier–Stokes equations without any scaling arguments and with minimal assumptions. This result is therefore the theoretical complement to experimental and numerical verifications of the $-5/3$ law (e.g. Sreenivasan 1995; Yeung & Zhou 1997).

By introducing a combination matrix $C^{\boldsymbol{pq},\boldsymbol{k}}$, an exact, tractable representation of nonlinear triad interactions is shown to be possible. Using this matrix in the governing equations, all of the interactions between two wavenumbers $\boldsymbol{p}$ and $\boldsymbol{q}$ can be conveniently accounted for, and the dependence on $\boldsymbol{k}$ is explicitly shown. Subsequently, the spectral energy equation is derived for homogeneous, isotropic turbulence. By casting the nonlinear terms in a canonical form, we obtain a representation of the nonlinear operator in Fourier space which is amenable to further manipulation. The outcome is the ability to express the nonlinear operator in terms of the energy density (4.8) rather than a convolution (4.7). With this representation, we derive the $-5/3$ decay rate of the energy wavenumber spectrum in the inertial range without any scaling arguments.

Future work can apply the concepts presented herein to various problems related to homogeneous isotropic turbulence. Examples include the behaviour of the energy spectra dissipation range; the importance of sweeping motions (Kraichnan 1964; Tennekes 1975) which do not appear in the energy spectra but could affect higher order spectra (Zhou *et al.* 2004; Zhou 2010); and turbulent scalar transport in homogeneous turbulence (Corrsin 1951).

Appendix A

In this appendix, we compute the eigenvalues $\alpha^{\boldsymbol{p}}_m$ of the matrix $[\mathcal{W}^{\boldsymbol{pq}}_{ij}]^{-1}\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$. Starting from the choice of the weight matrix given in § 4, we first need to determine the inverse of $\mathcal{W}^{\boldsymbol{pq}}_{ij}$. This is made possible through the identity

$$\sum_{\boldsymbol{q}} C^{\boldsymbol{pq},\boldsymbol{k}} C^{\boldsymbol{qr},\boldsymbol{k}} = \delta^{\boldsymbol{pr}} \tag{A 1}$$

for any arbitrary wavenumber vector $\boldsymbol{k}$. Using this identity, we can compute $\left[\mathcal{W}^{\boldsymbol{pq}}_{ij}\right]^{-1}$ as

$$\left[\mathcal{W}^{\boldsymbol{pq}}_{ij}\right]^{-1} = \left[pC^{\boldsymbol{pq},\boldsymbol{0}}\delta_{ij}\right]^{-1} = \frac{1}{p}C^{\boldsymbol{pq},\boldsymbol{0}}\delta_{ij}. \tag{A 2}$$

Next we examine the structure of the matrix $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} = \boldsymbol{\Gamma}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m} + \boldsymbol{\Gamma}^{\boldsymbol{qp},\boldsymbol{k}}_{ji,m}$. Without loss of generality, we choose to express $\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,m}$ in two dimensions and for $m = 1$; the results are equally valid for other choices as well.

$$\mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,1} = \begin{pmatrix} p_1 C^{\boldsymbol{pq},\boldsymbol{k}} & 0 \\ p_2 C^{\boldsymbol{pq},\boldsymbol{k}} & 0 \end{pmatrix} + \begin{pmatrix} p_1 C^{\boldsymbol{pq},\boldsymbol{k}} & p_2 C^{\boldsymbol{pq},\boldsymbol{k}} \\ 0 & 0 \end{pmatrix} = C^{\boldsymbol{pq},\boldsymbol{k}} \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 0 \end{pmatrix} \tag{A 3}$$

Multiplying (A 2) by (A 3) gives

$$\left[\mathcal{W}^{\boldsymbol{pq}}_{ij}\right]^{-1} \mathcal{A}^{\boldsymbol{pq},\boldsymbol{k}}_{ij,1} = \frac{\boldsymbol{I}}{p} C^{\boldsymbol{pq},\boldsymbol{0}} C^{\boldsymbol{pq},\boldsymbol{k}} \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 0 \end{pmatrix}. \tag{A 4}$$

To find the eigenvalues α of the above matrix, we seek solutions where the following determinant vanishes:

$$\left| \frac{C^{\boldsymbol{pq},\boldsymbol{0}} C^{\boldsymbol{pq},\boldsymbol{k}}}{p} \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 0 \end{pmatrix} - \begin{pmatrix} \alpha\delta^{\boldsymbol{pq}} & 0 \\ 0 & \alpha\delta^{\boldsymbol{pq}} \end{pmatrix} \right| = 0. \tag{A 5}$$

We can use (A 1) along with the promotion operator to rewrite $C^{pq,\mathbf{0}}C^{pq,k} = (\boldsymbol{\mathcal{P}})^k \delta^{pq}$, and the property of the promotion and demotion operator given in § 2 allows (A 5) to be written as

$$\left| \frac{C^{pq,\mathbf{0}}C^{pq,k}}{p} \begin{pmatrix} 2p_1 & p_2 \\ p_2 & 0 \end{pmatrix} - \begin{pmatrix} \alpha(\boldsymbol{\mathcal{P}})^k(\boldsymbol{\mathcal{D}})^k & 0 \\ 0 & \alpha(\boldsymbol{\mathcal{P}})^k(\boldsymbol{\mathcal{D}})^k \end{pmatrix} \right| = \left|(\boldsymbol{\mathcal{P}})^k\right| \left| \begin{pmatrix} 2p_1\boldsymbol{I} - \alpha q(\boldsymbol{\mathcal{D}})^k & p_2\boldsymbol{I} \\ p_2\boldsymbol{I} & -\alpha p(\boldsymbol{\mathcal{D}})^k \end{pmatrix} \right| = 0. \tag{A 6}$$

Computing the block-matrix determinant and manipulating the algebra gives the result that $\left|2p_1 p\alpha(\boldsymbol{\mathcal{D}})^k - \alpha^2 p(\boldsymbol{\mathcal{D}})^{2k} + p_2^2\boldsymbol{I}\right| = 0$. Here we express powers of the demotion operator $(\boldsymbol{\mathcal{D}})^k$ in terms of the diagonal matrix $\boldsymbol{\Lambda}$ through $(\boldsymbol{\mathcal{D}})^k = \boldsymbol{\Sigma}\,(\boldsymbol{\Lambda})^k\,(\boldsymbol{\Sigma})^{-1}$, and inserting into the above result gives

$$\left|\Sigma\left(2p_1 p\alpha(\boldsymbol{\Lambda})^k - \alpha^2 p(\boldsymbol{\Lambda})^{2k} + p_2^2(\boldsymbol{\Lambda})^0\right)\Sigma^{-1}\right| = 0. \tag{A 7}$$

For this equation to hold, we find that every diagonal term must be zero:

$$2p_1 p\alpha \mathrm{e}^{\mathrm{i}k\theta_i} - \alpha^2 p^2 \mathrm{e}^{2\mathrm{i}k\theta_i} + p_2^2 = 0, \tag{A 8}$$

where $\mathrm{e}^{k\theta_i}$ are the diagonal elements of $\boldsymbol{\Lambda}$. If we substitute the variable $\tilde{\alpha} = \alpha\mathrm{e}^{k\theta_i}$ and solve the quadratic equation for $\tilde{\alpha}$, we find

$$\tilde{\alpha}_1^{\boldsymbol{p}} = \frac{1}{p}\left[p_1 \pm \sqrt{p_1^2 + p_2^2}\right] = \frac{p_1}{p} \pm 1, \tag{A 9}$$

which has a maximum value of $\max\left\{\tilde{\alpha}_1^{\boldsymbol{p}}\right\} = 2$. In (A 9), $\tilde{\alpha}_1^{\boldsymbol{p}}$ is composed of two parts: a directional component $\tilde{\alpha}' = p_1/p$, which depends on $\boldsymbol{p}$, and an averaged component $\overline{\tilde{\alpha}_1} = \pm 1$, which is independent of $\boldsymbol{p}$. When (A 9) is inserted into (4.9), the only surviving component is $\overline{\tilde{\alpha}_1}$, due to the fact that $\hat{\mathcal{E}}(\boldsymbol{k}) = \hat{\mathcal{E}}(-\boldsymbol{k})$. Using this fact, along with the observation that $\overline{\tilde{\alpha}}_m$ is independent of the choice of m, allows (4.9) to be rewritten as

$$\frac{\partial}{\partial t}\hat{\mathcal{E}}(\boldsymbol{k}) + 2\nu k^2 \hat{\mathcal{E}}(\boldsymbol{k}) = -\mathrm{i}\sum_m \overline{\alpha}\left(\sum_{\boldsymbol{p}} p\hat{\mathcal{E}}(\boldsymbol{p})\right)\hat{u}_m^{\boldsymbol{k}}. \tag{A 10}$$

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