Minimizing Interactions in Mixed Oscillator Networks

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Abstract—This paper investigates the use of a single control node to minimize the interactions in a network of coupled oscillators as it maintains a synchronous state in the face of disturbances. We define an $\mathcal{H}_2$-norm that quantifies these interactions by considering the effect of exogenous disturbances on the relative separation of oscillator phases. Under some simplifying assumptions, we show that a single control node that is directly coupled to every oscillator cannot reduce the total network interactions. Additionally, we analyze the system in the frequency domain to gain insight about the interactions in such a network.

I. INTRODUCTION

Systems of coupled oscillators have a diverse set of applications ranging from power grids to biological systems [1]. One important phenomenon that is often studied in these types of systems is synchronization, a state in which all of the oscillators have reached some mutual goal, e.g. when the rotor phase angle and frequencies of a system of swing equations are aligned [2].

Much of the work in this area focuses on determining when a system will synchronize. However, once it is known that the system will return to synchrony, one may also be interested in the performance of the system. Performance measures such as robustness of a synchronous state for systems of coupled oscillators have been investigated using an $\mathcal{H}_2$-norm from disturbances at each oscillator to a particular output whose Euclidean norm is a measure of how close the oscillators are to synchrony [3], [4]. An $\mathcal{H}_2$-norm can be used to investigate coherence, i.e. how well the oscillators act like a single unit, in for example, vehicle formations [5], [6]. Alternatively, the output of a system can be defined such that the system’s $\mathcal{H}_2$-norm quantifies the total system interactions required to reach synchrony. Bamieh and Gayme [7], for example, study power system dynamics using a linear model of a Kron-reduced network. They define the system output such that the square of the $\mathcal{H}_2$-norm measures the real power loss transients incurred in maintaining synchronization in the face of disturbances, which corresponds to measuring interactions in the form of power transfer between generators.

A number of works focus on improving the synchronization performance of coupled oscillators using feedback control. For example, Dörfler et al. investigate control strategies to optimize the coherence of power grids [8]. The synchronization performance of a heterogeneous network of coupled first and second order dynamical systems can be quantified through an $\mathcal{H}_2$-norm similar to that defined by Bamieh and Gayme [7]. An example of such a system is the linearized structure preserving power grid model introduced by Bergen and Hill [9]. In that work the second order oscillators model synchronous generators and the first order oscillators model frequency dependent loads. This framework has also been adapted to the analysis of systems with induction motor loads [10] and power grids with renewable power generation [11].

In this paper we consider a linear network of coupled oscillators with a mixture of first and second order dynamics connected over a graph. We refer to the subsystems as nodes, and assume that we can select the parameters of the first order nodes in order to shape the dynamics of the network as a whole. This is physically motivated by systems such as power grids where the parameters of the second order nodes are fixed, but the parameters of the first order nodes are easily adjusted through power electronics. An application for this approach is the design of a droop controller for a storage device or renewable energy source being added to an existing grid, where the goal is to minimize the power flow signals used to maintain synchrony. Our problem setting and result is related to investigations that assess the controllability of a network of coupled systems with a subset of nodes designated as control inputs or “leaders.” The basic question asked in those works is whether the entire network is controllable from the leaders [12], [13], [14].

The specific problem we consider is that of minimizing the $\mathcal{H}_2$-norm from disturbances acting on the second order nodes to an output that provides a measure of the total interactions that must occur between the nodes during resynchronization. We thus define the output so that its squared Euclidean norm is the weighted sum of squares of the phase differences between connected nodes. These connections and the corresponding weights are given by an output graph. As a preliminary step, we investigate the special case of a single first order node, which may, for example, correspond to adding a single droop controlled device to a conventional power grid of synchronous generators. In particular, we consider a network of $k$ nodes with homogeneous second order oscillators at $k − 1$ nodes. The remaining node (the control node) has first order dynamics. When the control node has degree $k − 1$, we prove, under some technical conditions, that the optimal system performance is obtained when the control node instantaneously tracks the arithmetic mean of the second order nodes.

The remainder of this paper is organized as follows. In Section II we introduce the mathematical formulation for the control node optimization problem. Our main result is presented in Section III. In Section III we also provide closed
form expressions for the singular values of our system under simplifying assumptions in order to provide a more intuitive interpretation of the main result. In Section IV we present numerical examples in order to provide insight into what can happen when the degree of the control node is less than $k-1$.

II. PROBLEM FORMULATION

In this section we introduce the mathematical notation used in this paper. Then we present the system model and a related optimal control problem that will be useful in the proof of our main result.

A. Notation and Mathematical Preliminaries

We denote scalar parameters, $a \in \mathbb{R}$, by lowercase letters, vectors, $v \in \mathbb{R}^n$, by bold lowercase letters, and matrices, $M \in \mathbb{R}^{m \times n}$, by uppercase letters. $[M]_{i,j}$ and $[v]_i$ respectively denote the element in the $i$th row and $j$th column of a matrix, and the $i$th element of a vector. $\|v\|_2$ denotes the Euclidean norm of $v \in \mathbb{R}^n$. Given a matrix, $M \in \mathbb{R}^{m \times n}$, we denote its transpose, complex conjugate transpose by $M^T$, $\overline{M}$ and $M^*$ respectively. Given a set, $\mathcal{S}$, we denote its cardinality by $|\mathcal{S}|$.

We define a weighted graph, $\mathcal{G}$, as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V}$ is the set of vertices, $\mathcal{E}$ is the set of edges which are unordered pairs of vertices, and $\mathcal{W}: \mathcal{E} \rightarrow \mathbb{R}_+$ is a mapping from the edges to the positive real numbers, called the weight of each edge. We assume that $\mathcal{V}$ is a set of cardinal numbers, i.e. $\mathcal{V} = \{1, \ldots, k\}$ for some $k \in \mathbb{Z}_+$. We use the notation $i \sim j$ to mean that $\{i, j\} \in \mathcal{E}$ and refer to $i$ and $j$ as being adjacent vertices. $L_\mathcal{E} \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$ is the weighted Laplacian of a graph, $\mathcal{G}$, is defined by $L_\mathcal{E} = D_\mathcal{V} - A_\mathcal{E}$ where $A_\mathcal{E}$ is the weighted adjacency matrix defined by $A_{i,j} = w_{i,j}$ if $\{i, j\} \in \mathcal{E}$, and $A_{i,j} = 0$ otherwise. $D_\mathcal{V}$ is diagonal, with $D_{\mathcal{V}}_{i,i} = \sum_{j \sim i} w_{i,j}$. A graph, $\mathcal{G}$, is complete if $\{i, j\} \in \mathcal{E}$, $\forall i \leq j \leq |\mathcal{E}|$.

B. Problem Setting

We consider a system composed of $k$ nodes (subsystems) coupled across a connected weighted graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E}_a, \mathcal{W}_b)$, with each vertex corresponding to a node. Let $q$ and $r$ be nonnegative integers such that $q = q + r$, and partition the vertex set $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, with $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. We order the elements of $\mathcal{V}$ such that $\mathcal{V}_1 = \{1, \ldots, q\}$, and the remaining $r$ vertices are in $\mathcal{V}_2$. The nodes associated with the vertices in $\mathcal{V}_1$ have first order dynamics, and the nodes associated with vertices in $\mathcal{V}_2$ have second order dynamics. In what follows, we freely refer to nodes by the element of $\mathcal{V}$ that they are associated with, e.g. “the $i$th node” refers to the node associated with $i \in \mathcal{V}$.

The dynamics at each node $i \in \mathcal{V}$ are given by

$$m_i \ddot{\theta}_i + \zeta_i \dot{\theta}_i + \sum_{j \sim i} b_{i,j} (\theta_i - \theta_j) = 0$$

where $\theta_i$, $m_i$, and $\zeta_i$ are respectively the phase, inertia, and damping of the $i$th node. We assume that $m_i > 0$, $\forall i \in \mathcal{V}_1$, $m_i > 0$, $\forall i \in \mathcal{V}_2$, $\zeta_i > 0$, $\forall i \in \mathcal{V}_1$, and $\zeta_i > 0$, $\forall i \in \mathcal{V}_2$. If $\{i, j\} \in \mathcal{E}_a$, then $b_{i,j} = \mathcal{W}_b(\{i, j\})$, otherwise, $b_{i,j} = 0$.

Remark 1: By abuse of terminology, when $m_i = 0$ we refer to node $i$ as a “first order oscillator,” although it does not have imaginary eigenvalues.

We define synchrony in this system as the state in which $\theta_i = \theta_j$, $\forall i, j \in \mathcal{V}$ and $\theta_i = 0$, $\forall i \in \mathcal{V}_2$. We assume that this system is stable (i.e. will return to a synchronous state) and investigate its behavior under small disturbances.

Remark 2: The synchronous state could correspond to an arbitrary operating point through a change of variables.

In order to analyze the input-output behavior of the system we apply an exogenous input, $u_i$, to each second order node, so that the dynamics at each node are given by

$$m_i \ddot{\theta}_i + \zeta_i \dot{\theta}_i + \sum_{j \sim i} b_{i,j} (\theta_i - \theta_j) = m_i u_i$$

and consider the performance measure

$$P(t) = \sum_{i \sim j} g_{i,j} (\theta_i - \theta_j)^2,$$

where the weights $g_{i,j}$ are determined by the output graph, $\mathcal{O} = (\mathcal{V}, \mathcal{E}_g, \mathcal{W}_g)$, as follows. $g_{i,j} = \mathcal{W}_g(\{i, j\})$ if $\{i, j\} \in \mathcal{E}_g$, $g_{i,j} = 0$ otherwise.

Remark 3: In general, we can allow $\mathcal{C}$ and $\mathcal{O}$ to have different edge sets, however since we want $P(t)$ to measure the amount of interaction between adjacent nodes we enforce $\mathcal{E}_w = \mathcal{E}_g$, so that whenever one node influences another, there is a corresponding penalty (contribution to $P(t)$).

We define the output of our system, $y(t)$, such that $P(t) = \|y(t)\|_2^2$, and call the system with input $u = [u_i]$ and output $y$, $G$. A state space realization of $G$ is then given by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, $A = \begin{bmatrix} -T_Z L_B & T_2 \\ -M^{-1} T_2^T L_B & -M^{-1} Z_2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}^T$, and $C = \begin{bmatrix} L_G^T/2 \\ 0 \end{bmatrix}$. $L_B$ and $L_G$ are the weighted Laplacians of $\mathcal{C}$ and $\mathcal{O}$ respectively. Also, $T_Z = \begin{bmatrix} Z_1 \zeta_1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k \times k}$, where $Z_1 = \text{diag}(\zeta_1, \ldots, \zeta_q)$. $T_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}^T \in \mathbb{R}^{r \times r}$, $Z_2 = \text{diag}(\zeta_{q+1}, \ldots, \zeta_k) \in \mathbb{R}^{r \times r}$, and $M = \text{diag}(m_{q+1}, \ldots, m_k)$.

The square of the $H_2$-norm of $G$, $\|G\|_2^2$, measures how much the nodes interact over edges while returning to synchrony given a disturbance $u$. We refer to this as the “cost” of maintaining synchrony.

The goal of this work is to determine the coefficients $[\zeta_1, \ldots, \zeta_k]$ that minimize $\|G\|_2^2$, i.e. solve the following optimization problem:

$$\min_{[\zeta_1, \ldots, \zeta_k]^T \in \mathbb{R}_{\geq 0}^k} \|G\|_2^2$$

Our problem thus amounts to choosing the parameters for a set, $\mathcal{V}_1$, of control nodes in order to minimize the “cost” of returning to synchrony after a disturbance.
C. A Related Optimal Control Problem

In this section we first introduce the dynamics of a new system which is similar to (3a) with \( q = 1 \), but has the first order oscillator replaced with an arbitrary linear system \( F \). Then we define an output equation for this new system whose 2-norm is equivalent to \( P(t) \) in (2). Finally, we observe that finding the \( F \) that minimizes the \( \mathcal{H}_2 \)-norm of this related system is the standard Linear Quadratic Regulator (LQR) synthesis problem in optimal control.

As before, we consider a system of coupled second order oscillators and control nodes. Here we assume that \( q = 1 \), i.e., there is only one control node. In contrast to the previous section, we relax the restriction on the dynamics of the control node and instead allow it to have any dynamics given by a linear system, \( F : \mathbb{C}^{k-1} \rightarrow \mathbb{C}^k \), such that

\[
F \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \theta_c.
\]

(5)

where \( \theta_2 = [\theta_2, \ldots, \theta_k]^T \) is the vector of \( k-1 \) phases associated with the second order nodes. For all nodes \( i \in \mathcal{V}_2 \) (the second order oscillators), equation (1) can be rewritten as

\[
m_i \dot{\theta}_i = - \sum_{i>0} b_{i,j} (\theta_i - \theta_j) - \zeta_i \dot{\theta}_i
- b_{1,i} (\theta_i - \theta_c(t)) + m_i u_i.
\]

(6)

Together, (5) and (6) describe the dynamics of a system with \( k-1 \) second order oscillators and a single control node whose dynamics are given by \( F \).

Let \( L_{B2} \) and \( L_{G2} \) be the respective graph Laplacians of the subgraphs of \( \mathcal{C} \) and \( \mathcal{O} \) induced by \( \mathcal{V}_2 \). Also, let \( b_c, g_c \in \mathbb{R}^{k \times 1} \) be defined by \( [b_j, j=1, \ldots, k] = \mathcal{W}_b \{ \{i,j\} \}, \forall \{i,j\} \in \mathcal{E}_b \) and \( [g_{j, j=1, \ldots, k}] = \mathcal{W}_g \{ \{i,j\} \}, \forall \{i,j\} \in \mathcal{E}_g \). If \( \{i,j\} \notin \mathcal{E}_b \) or \( \{i,j\} \notin \mathcal{E}_g \), the corresponding element of \( b_c \) or \( g_c \) is respectively zero. If we let \( B_c = \text{diag}(b_c) \) we can rewrite (6) as

\[
\begin{bmatrix} \dot{\theta}_2 \\ \theta_3 \end{bmatrix} = A_2 \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} + B_1 \theta_c + B_2 u,
\]

(7)

where

\[
A_2 = \begin{bmatrix} 0 & I \\ -M^{-1} (L_{B2} + B_c) & -M^{-1} Z_2 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 \\ M^{-1} b_c \end{bmatrix}, \quad \text{and} \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}.
\]

The following proposition gives an output equation for (7) such that the 2-norm of the output is equivalent to the performance measure \( P(t) \) in (2).

**Proposition 1:** Let \( y_c = C_2 \theta_2 + D_2 \theta_c \) where \( \begin{bmatrix} D_2 \\ C_2 \end{bmatrix} = I_{2,2} \). Then \( \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix}^T = \theta \) implies that

\[
P(t) = \| C_2 \theta_2 + D_2 \theta_c \|^2_2.
\]

**Proof:** Simply observe that \( C_2 \theta_2 + D_2 \theta_c = I_{2,2} \begin{bmatrix} \theta_c \\ \theta_2 \end{bmatrix} \), and so we have

\[
\| C_2 \theta_2 + D_2 \theta_c \|^2_2 = \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix}^T L_2 \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \theta^T L_2 \theta = P(t).
\]

When \( G_c : u \rightarrow y_c \) we have an expression for a performance measure of the form (2) in the sense that when the system \( F \) corresponds to a first order oscillator, \( \| G_c \|^2_2 = \| G \|^2_2 \).

A problem statement analogous to that in (4) can now be formulated in terms of finding the system \( F \) that minimizes the \( \mathcal{H}_2 \)-norm of \( G_c \). It is a standard fact [15, p. 406] that this problem is equivalent to synthesizing an infinite horizon LQR using the cost functional

\[
J = \int_0^\infty (x_2^T Q x_2 + R \theta_c^2 + 2x_2^T N \theta_c) dt,
\]

(8)

where \( x_2 = [\theta_1^T, \theta_2^T]^T \) is the trajectory due to arbitrary initial conditions and input \( \theta_c \), \( Q = \begin{bmatrix} L_{G2} + \text{diag}(g_c) & 0 \\ 0 & 0 \end{bmatrix} \), \( R = \sum_{i=1}^{k-1} [g_c]_i \), and \( N = \begin{bmatrix} -g_c^T & 0 \end{bmatrix} \). In addition, \( F \) will be a static gain matrix [16].

We can now state our optimal control problem as follows:

\[
\min_{F \in \mathbb{R}^{1 \times r}} \| G_c \|^2_2,
\]

which can be rewritten as

\[
\min_{F \in \mathbb{R}^{1 \times r}} J
\]

subject to \( x_2 = A_2 x_2 + B_1 \theta_c \).

**Remark 4:** Although this paper concentrates on the special case when \( F \) has first order dynamics, (9) provides some insight into the relationship between the problem of optimizing control node dynamics and traditional optimal control problems.

III. ANALYTICAL RESULTS

In this section we consider the system \( G \) defined by (3) with \( q = 1 \), i.e. only one control node, and investigate the solution to (4) for the special case when the control node is connected to all of the second order nodes. In the next two subsections we first prove that under certain conditions the optimal value of \( \zeta_1 \) is zero. We then directly compute the singular values of the system under additional conditions in order to provide insight as to how the first and second order nodes interact in this special case.

A. The Optimal Damping Strategy

The main result of this section states that the optimal phase trajectory \( \theta_c(t) \) of a single control node that is connected to all of the second order nodes instantaneously tracks the average phase of the second order nodes. This notion is formalized in the following theorem.

**Theorem 1 (Main Result):** Consider the system \( G \) defined by (3) with \( q = 1 \) under the following conditions:

1) \( \zeta_1 = \zeta \), \( m_i = m \forall i \in \mathcal{V}_2 \) (i.e. the second order nodes have uniform inertia and damping.)

2) \( \{1,i\} \in \mathcal{E}, \forall i \in \mathcal{V}_2 \) (i.e. the control node is connected to all of the second order nodes.)

3) \( \mathcal{W}_b \{ \{1,i\} \} = b_c, \mathcal{W}_g \{ \{1,i\} \} = g_c, \forall i \in \mathcal{V}_2 \) (i.e. the coupling and output weights of the edges connecting the control node to the second order nodes are uniform.)
The nonnegative damping at the control node, $\zeta_1$, that minimizes $\|G\|_2^2$, is zero.

Proof: Observe that because $q = 1$, we can use the problem formulation from Section II-C. We will solve (9) and observe that the optimal controller is in fact a first order oscillator with $\zeta_1 = 0$. The controllable subspace of $(A_2, B_1)$ in (7) is given by $S = \begin{bmatrix} a_{1, k - 1} \\ b_{1, k - 1} \end{bmatrix}$, where $a, b \in \mathbb{R}$. Let $v \in S^\perp$, then it can be shown that under our assumptions, $z_S^T A_2 v = 0, \forall z_S \in S$, and hence $S^\perp$ is $A_2$-invariant and thus invariant with respect to (7).

Denote the state vector of (7) by $z = z_S + z_{S^\perp}$, where $z_S = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, and observe that the optimal controller is in fact a first order oscillator, and hence we cannot use the proof method that was applied to Theorem 1 above to find the optimal value of $\zeta_1$.

Theorem 1 implies that when the control node is connected to all of the second order nodes through a graph that satisfies the conditions of Theorem 1, the optimal control node has infinite bandwidth, in the sense that it tracks the mean phase of the system of oscillators instantaneously, and therefore does not damp their motion. In the next section we perform a frequency domain analysis for a special case of the problem that provides a more restrictive version of Theorem 1. This simplified problem setting provides some additional insight regarding the results of Theorem 1.

B. Singular Values

The $H_2$-norm of $G$ defined by (3) can be computed from

$$\|G\|_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i} \sigma_i(j\omega)^2 d\omega,$$

where $\sigma_i(j\omega)$ is the $i$th singular value of the frequency response of $G$, $G(j\omega)$. Therefore the effect of $\zeta_1$ on the $H_2$-norm of $G$ can be examined by finding the dependence of $\sigma_i(j\omega)$ on $\zeta_1$. In this section we construct closed form expressions for the singular values of $G(j\omega)$, to provide insight into how the control node affects the second order oscillators. In order to do this we further restrict the conditions of Theorem 1 by requiring that $C$ and $O$ be complete graphs with uniform edge weights. We treat the cases $\zeta_1 > 0$ and $\zeta_1 = 0$ separately.

1) Nonzero Damping at the Control Node ($\zeta_1 > 0$): We first compute a closed form expression for $G(j\omega)$, and then we compute its singular values.

Lemma 1: Consider the system $G$ defined by (3) with the assumptions of Theorem 1 as well as

1) $\zeta_1 > 0$ (The control node has nonzero damping.)
2) $W_b \{i, j\} = b, W_g \{i, j\} = g, \forall 1 \leq i, j \leq k$ (The coupling and output graphs, $C$ and $O$, are both complete with uniform edge weights.\footnote{Note that we have implicitly assumed that $\{i, j\} \in \mathcal{E}, \forall 1 \leq i, j \leq k.$})

The transfer matrix of $G$ is given by

$$G(s) = \begin{bmatrix} h_{r1}(s) & h_{r1}(s) & \cdots & h_{r1}(s) \\ h_{sd}(s) & h_{sd}(s) & \cdots & h_{sd}(s) \\ \vdots & \vdots & \ddots & \vdots \\ h_{od}(s) & h_{od}(s) & \cdots & h_{sd}(s) \end{bmatrix}$$

where

- $h_{r1}(s) = \frac{-\zeta_1 m \sqrt{g}}{\sqrt{k} p_2(s)}$
- $h_{sd}(s) = \frac{m \sqrt{g} p_3(s)}{\sqrt{k} p_1(s) p_2(s)}$
- $h_{od}(s) = \frac{-m \sqrt{g} (ms + \zeta) (\zeta_1 s + fb)}{\sqrt{k} p_1(s) p_2(s)}$
and \( p_1(s) = ms^2 + \zeta s + kb \),
\[
p_2(s) = \zeta_1 ms^2 + [\zeta_1 + b(k-1)m]s + (k-1)bk + \zeta_1 b,
\]
\[
p_3(s) = (k-1)\zeta_1 ms^2 + (k-1)\zeta_1 s + k(k-2)bm
\]
\[+ k\zeta_1 b + (k-2)b\zeta_1.\]

**Proof:** Due to space limitations we omit the full proof and instead provide an outline.

\[ G(s) = C(sI - A)^{-1}B \]

can be constructed by making use of the fact that \( (sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \)

and then computing the characteristic polynomial, \( p_A(s) \), and adjugate,\(^4\) \( \text{adj}(sI - A) \), separately. The Schur complement formula for the determinant [17, p. 40] can be used to show that

\[ p_A(s) = \frac{sp_1(s)^{k-2}p_2(s)}{m^2\zeta_1}. \tag{15} \]

If we partition \( \text{adj}(sI - A) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \) so that it is block conformal to \( A \) and \( C \), then it is clear from the structure of \( B \) and \( C \) that

\[ G(s) = \frac{1}{p_A(s)} L_{G1}^2 \bar{G} N_{12}. \tag{16} \]

The definition of the adjugate and the properties of the determinant can be used to show that \( N_{12}(s) \) has the proposed structure of \( G(s) \).

The completion of the proof simply requires computing only three of the elements of \( \text{adj}(sI - A) \), one corresponding to each of the three distinct entries of \( N_{12}(s) \). This is done in a straightforward manner by rearranging rows and/or columns of \( sI - A \), and using the Schur complement formula for the determinant. Then (16) is computed using the fact that for a complete graph with uniform edge weights, the determinant can be used to show that

\[ \sigma_1 = \frac{\zeta_1 m\sqrt{g}}{\sqrt{p_2(j\omega)p_2(-j\omega)}} \]
\[ \sigma_2 = \frac{m\sqrt{kg}}{\sqrt{p_1(j\omega)p_1(-j\omega)}} \]

where \( \sigma_1 \) has multiplicity one, and \( \sigma_2 \) has multiplicity \( k-2 \).

**Proof:** We proceed by simply presenting the left and right singular vectors of \( \sigma_1 \) as well as the \( k-2 \) left and right singular vectors of \( \sigma_2 \).

<table>
<thead>
<tr>
<th>( \sigma_1 )</th>
<th>( \sigma_2 )</th>
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| R. \[
\sqrt{\frac{p_2(-j\omega)}{k_{p_2(j\omega)}}} \] 1\( \times \)1, \( \|v_i\|_2 = 1 \] | \[
\sqrt{\frac{p_1(j\omega)p_1(-j\omega)}{p_2(j\omega)}} \] 0 \( \times \)1 |
| L. \[
\sqrt{\frac{p_2(-j\omega)}{k_{p_2(j\omega)}}} \] 1\( \times \)k-1, \( \|v_i\|_2 = 1 \] | \[
\sqrt{\frac{p_1(j\omega)p_1(-j\omega)}{p_2(j\omega)}} \] 0 \( \times \)1 |

^4 Also known as the classical adjoint.

In the table above, the “R.” and “L.” rows are the right and left singular vectors respectively. These values can easily be verified.

Looking at the singular values of \( G(j\omega) \) with \( \zeta_1 > 0 \), we can see that \( \sigma_1 \) corresponds to the second order oscillators moving relative to the control node, while \( \sigma_2 \) corresponds to the second order oscillators moving relative to one another. In fact, in the next section we will see that when \( \zeta_1 = 0 \), the only change in the singular values of \( G(j\omega) \) is that \( \sigma_1 \) is removed.

2) *Zero Damping at the Control Node (\( \zeta_1 = 0 \)):* In this case the dynamics at the control node are given by an algebraic equation, therefore (3) is not a state space realization for \( G \). In order to obtain a state space realization we must remove this algebraic constraint (i.e. eliminate \( \theta_1 \) from our system of equations). Specifically, we construct a realization for a system, \( \tilde{G} \), whose frequency response, \( G(j\omega) \), has the same singular values as \( G(j\omega) \). We then proceed as in the case \( \zeta_1 > 0 \), by first constructing the transfer matrix, \( G(s) \), and then the singular values of \( \tilde{G}(j\omega) \).

**Lemma 2:** Consider the system \( G \) defined by (3), with the same assumptions as in Lemma 1, except that instead of \( \zeta_1 > 0 \), we have \( \zeta_1 = 0 \). Under these conditions, the frequency response of \( G \) has the same singular values as that of \( G \), where \( G \) is given by the following realization:

\[ \tilde{G} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}. \tag{18} \]
\[ \tilde{A} = \begin{bmatrix} 0 & I \\ -M^{-1}\tilde{L}_B & -M^{-1}\tilde{Z}_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \text{and } \tilde{C} = \begin{bmatrix} \tilde{L}_G^T \\ 0 \end{bmatrix}, \]

where \( \tilde{L}_B \) and \( \tilde{L}_G \) are the weighted Laplacians of complete graphs of size \( k-1 \), with respective edge weights \( \frac{kb}{k-1} \) and \( \frac{kb}{k-1} \).

**Proof:** When \( \zeta_1 = 0 \), \( G \) is described by

\[ 0 = \zeta_1 \dot{\theta}_1 = -(k-1)bk_1 + \sum_{i=2}^{k} \theta_i, \tag{19} \]
\[ \dot{\theta}_i = \frac{1}{m} \left( -(k-1)bk_1 + \sum_{j=2, j \neq i}^{k} \theta_j \right) + bk_1 - \zeta_1 \dot{\theta}_1 + u_i, \tag{20} \]

Solving equation (19) for \( \theta_1 \) and substituting it into equation (20) yields

\[ \dot{\theta}_i = \frac{1}{m} \left( \frac{-kb}{k-1} (k-2) \theta_i + \frac{kb}{k-1} \sum_{j=2, j \neq i}^{k} \theta_j - \zeta_1 \dot{\theta}_1 + u_i \right). \]

This is the set of equations that describe a system of \( k-1 \) second order nodes, with a complete coupling graph, \( C \), that has uniform edge weights \( \frac{kb}{k-1} \).

Let \( T_1 = \begin{bmatrix} 1 & 1^T \end{bmatrix} \in \mathbb{R}^{k \times k-1} \). Then \( \theta = T_1 \theta_2 \), and hence

\[ x = Tx_2 \text{ where } T = \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix}. \]

Therefore, when \( \zeta_1 = 0 \),
\[
\begin{pmatrix}
\tilde{A} & \tilde{B} \\
CT & 0
\end{pmatrix}
\] is a realization of \( G \), and so \( G^*(j\omega) G(j\omega) = B^T \left( j\omega I - \tilde{A} \right)^{-1} T^\ast C^T \left( j\omega I - A \right)^{-1} \tilde{B} \). It can be shown that \( T^\ast C^T = \tilde{C}^T \tilde{C} \), and so we have that \( G^*(j\omega) G(j\omega) = \tilde{G}^*(j\omega) \tilde{G}(j\omega) \). Since the singular values of \( G(j\omega) \) are the eigenvalues of \( G^*(j\omega) G(j\omega) \), we have shown that \( \tilde{G}(j\omega) \) has the same singular values as \( G(j\omega) \).

The next lemma gives a method of constructing \( \tilde{G}(s) \).

**Lemma 3:** Given the system \( G \), with the same assumptions as in Lemma 2, the transfer matrix, \( \tilde{G}(s) \), is given by

\[
\tilde{G}(s) = \frac{m \sqrt{k_g}}{(k-1)(ms^2 + \zeta s + kb)} L_{\text{complete}}
\]

where \( L_{\text{complete}} \) is the unweighted graph Laplacian of a complete graph with \( k - 1 \) vertices.

**Proof:** The proof is very similar to that of Lemma 1, therefore we omit the details due to space constraints.

The singular values of the transfer matrix \( \tilde{G}(s) \) can be computed in a manner similar to the case when \( \zeta_1 > 0 \).

**Theorem 3:** Under the assumptions of Lemma 2, the sole singular value of \( G(j\omega) \) is

\[
\sigma_1 = \frac{m \sqrt{k_g}}{\sqrt{p_1(j\omega)p_1(-j\omega)}}
\]

**Proof:** By Lemma 2 the singular values of \( G(j\omega) \) and \( \tilde{G}(j\omega) \) are the same. Therefore we show that the given singular value is the only singular value of \( \tilde{G}(j\omega) \). Let \( S = \{ x \in \mathbb{R}^{k-1} | x^\top 1 = 0 \} \). \( S \) is the eigenspace of \( \tilde{G}(j\omega) \) associated with eigenvalue

\[
\lambda = \frac{m \sqrt{k_g}}{(ms^2 + \zeta s + kb)}
\]

because \( \tilde{G}(j\omega) \) is the Laplacian of a complete graph with uniform edge weights. Let \( v \in S \), then since \( \tilde{G}(j\omega) \) is symmetric, \( G^*(j\omega) \tilde{G}(j\omega) v = \tilde{G}(j\omega) G(j\omega) \tilde{G}(j\omega) v = \lambda v \). Therefore \( \sqrt{\lambda} = \frac{m \sqrt{k_g}}{\sqrt{p_1(j\omega)p_1(-j\omega)}} \) is a singular value of \( \tilde{G}(j\omega) \) with multiplicity \( \dim S = k - 2 \).

If \( w \in S^\perp, G^*(j\omega) \tilde{G}(j\omega) w = 0 \), so \( \tilde{G}(j\omega) \) has no other singular values.

We have shown that the only effect of making \( \zeta_1 = 0 \) on the singular values of \( G(j\omega) \) is to remove one of the two distinct singular values that are present when \( \zeta_1 > 0 \), without altering the multiplicity of the remaining singular value. Therefore it is immediately apparent from (13) that the \( H_2 \)-norm will be minimized when \( \zeta_1 = 0 \).

In the special case considered in this section, additional damping at the control node can only affect input output pairs associated with inputs in the 1 direction. This mode, which corresponds to equal changes to the phases of every node, is unobservable when \( \zeta_1 = 0 \).

**IV. NUMERICAL STUDIES**

In this section we present numerical case studies to both validate the theory developed in Section III and to explore the effects of changing the interconnection topology. We consider coupling graphs with uniform edge weights and the following four different topologies:

Case 1) A line graph. (See Fig. 1a for an example.)

Case 2) A circle graph. (See Fig. 1b for an example.)

Case 3) A complete graph of second order nodes connected to a control node of degree one. (See Fig. 1c for an example.)

Case 4) A complete graph.

For all but one topology, the degree of the control node is less than \( k - 1 \), and thus our result that the optimal value of \( \zeta_1 \) is zero does not hold. The output graphs have the same topology as the coupling graphs, and also have uniform edge weights. For each topology we solve (4) numerically for networks of size \( k = 2 \) to 100.

In this example we assume our system is a linearization of some nonlinear power grid model about a stable operating point. Therefore we pick system parameters that match real world power grids as follows: \( m = 0.0531, \zeta = 0.0265, b = 4.38, \) and \( g = 1.61 \). These inertia and damping values are based on parameters used in Sauer and Pai [18].

As shown in Fig. 2, the optimal \( \zeta_1 \) value for the complete graph (Case 4) is zero for all network sizes. This result is
predicted by the theory of Section III, since in this case the degree of the control node is $k - 1$. In the other cases, the optimal $\zeta_1$ value is nonzero for all $k > 2$, meaning a control node can reduce the total nodal interactions in the system.

One can think of adding a control node of the form $\zeta_1 \theta_c = - \sum b_{i,1} (\theta_c - \theta_i)$ as adding damping to the interconnected system. Therefore our results suggest a trade-off between the value of this added damping and the additional interactions that arise due to additional edges in the graph.

V. CONCLUSION

In this paper we consider a system of coupled second order oscillators connected to a set of first order control nodes. We define an input-output system such that its $H_2$-norm quantifies the total interactions between all of the oscillators as well as the oscillators and the control nodes. We investigate minimizing this norm by tuning the parameters of the control nodes, which is equivalent to designing an optimal controller with structural constraints. When there is a single control node and it is uniformly coupled to all of the second order nodes, the optimal control corresponds to the control node instantaneously tracking the mean phase of the second order nodes. This is a degenerate case of first order dynamics. Therefore the optimal closed loop system has the structure of the original system and we can obtain the solution to our original optimization problem by solving a related LQR synthesis problem. This standard LQR synthesis problem arises through a relaxation of the requirement that the control node have first order dynamics.

The numerical case studies validate the theory and illustrate interconnection topologies for which the optimal $\zeta_1$ value is nonzero. In these cases the solution to the LQR does not correspond to first order oscillator dynamics at the control node. An interesting open question is whether there exist systems for which the optimal $\zeta_1$ value is zero, but the LQR solution does not correspond to first order oscillator dynamics at the control node. The numerical case studies also show interesting asymptotic behavior of the optimal damping as the number of nodes grows, which is the topic of ongoing work.

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