Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination—Fall Semester
Real Analysis

Tuesday, August 19, 2025

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. If you complete the exam, you may leave before time is up, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

8. **If you leave the room you must leave behind all electronic devices–including phones.**

1. Define $x_1 = 2$ and for $n \geq 1$,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right).$$

   (a) Prove that the sequence $\{x_n\}$ converges.

   (b) Show that its limit is $\sqrt{3}$.

   (c) Establish the quadratic error bound

$$\left|x_{n+1} - \sqrt{3}\right| \leq \frac{1}{2\sqrt{3}} \left|x_n - \sqrt{3}\right|^2.$$

2. On $\mathbb{R}$ define

$$f_n(x) = \frac{n|x|}{1 + n^2x^2}.$$

   (a) Prove that $f_n \to 0$ pointwise on $\mathbb{R}$. Is this convergence uniform?

   (b) Determine whether the family $\{f_n\}$ is equicontinuous on $\mathbb{R}$.

3. Suppose f is continuous on $[0, 1]$, differentiable on $(0, 1)$, and satisfies $f(0) = f(1)$. Prove that there exists $c \in (0, 1)$ such that

$$f'(c) + f'(1 - c) = 0.$$

4. Consider the improper integral below parameterized by some $\alpha \in \mathbb{R}$

$$I(\alpha) = \int_0^1 x^\alpha \sin\bigl(1/x\bigr)\, dx.$$

   For what real values of α is this integral absolutely convergent? Justify your answer.

5. Consider the function

$$f(x) = \log(e^x + e^{-x}).$$

   (a) Prove that f is uniformly Lipschitz on all of $\mathbb{R}$.
   (That is, show for some constant $L > 0$, all x, y have $|f(x) - f(y)| \leq L|x - y|$.)

   (b) Prove that the nth derivative of f is also uniformly Lipschitz on $\mathbb{R}$.

(It may be useful in arranging your calculations to use the definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}.$$

Recall the calculus rules $\tanh' x = \operatorname{sech}^2 x$ and $\operatorname{sech}' x = -\operatorname{sech} x \tanh x$, and note both $|\tanh x| \leq 1$, $|\operatorname{sech} x| \leq 1$ for every $x \in \mathbb{R}$.)

6. Let G be an open subset of $\mathbb{R}$ and let $y \in G$ be arbitrarily chosen. Prove:

   (a) There exists a largest open interval I_y containing y.

   (b) If x and y are any two points in G, then either $I_x = I_y$ or $I_x \cap I_y = \varnothing$.

Department of Applied Mathematics and Statistics
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Probability

Wednesday, August 20, 2025

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. If you complete the exam, you may leave before time is up, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

8. **If you leave the room you must leave behind all electronic devices–including phones.**

1. Suppose $X_1, \ldots, X_n$ are iid Bernoulli random variables with parameter $p_n = c/n$ where $c \in (0, \infty)$. Find a simple expression for

$$\lim_{n\to\infty} E[\sum_{i=1}^{n} X_i^3].$$

2. A tortoise and rabbit agree to a one-mile race. The tortoise travels at a fixed velocity T, where T is a (nonnegative) random variable. The fixed velocity of the rabbit is a random variable $R = 2T - C$, where C and T are independent. We assume $C \sim \exp(1)$ and $T \sim \exp(\frac{1}{2})$ and that T and R are in units of miles-per-hour. Compute the probability the tortoise beats the rabbit.

   FYI: $X \sim \exp(\lambda)$ means the CDF of X is $F(x) = 1 - e^{-\lambda x}$ for $x > 0$.

3. A bag contains 1 blue ball, 4 red balls, and 2 yellow balls. You draw 4 balls from the bag without replacement. What is the probability mass function of the number of blue balls drawn conditioned on exactly 1 yellow ball being drawn?

4. Suppose X and Y are independent standard normal random variables, i.e., each has PDF

$$\varphi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \text{ for } -\infty < z < \infty.$$

   Now suppose $U = \frac{1}{2}(X+Y)$ and $V = \frac{1}{2}(X-Y)$. Use the Jacobian method to find the joint PDF of U and V. Also, say as much as you can about the joint and marginal distributions of U and V.

5. We have n balls ($n > 1$ fixed) and n distinct boxes. Each ball is placed into a box uniformly at random until all balls have been distributed. [Note: it is possible for a box to be occupied by more than one ball.] Compute the probability that there is at least one box that is unoccupied.

6. Let $\alpha > 0$. Let $X_1, X_2, X_3, \ldots$ be iid random variables with probability density

$$f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}}, & x \geq 1 \\ 0, & x \leq 1. \end{cases}$$

Let $S_n = X_1 + \cdots + X_n$. Determine for which $\alpha > 0$ (if any) there exists constants a and b for which

$$\frac{S_n - an}{b\sqrt{n}}$$

converges in distribution to a standard normal, and determine the values of a and b if they exist. Justify your assertions.

Department of Applied Mathematics and Statistics
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Introductory Examination–Spring Semester
Linear Algebra

August 21, 2025

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. If you complete the exam, you may leave before time is up, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**
8. **If you leave the room you must leave behind all electronic devices–including phones.**

1. Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^m$ such that $ABA = A$ and $\text{rank}[A|b] = \text{rank}A$. Prove that the linear system $Ax = b$ has solution $\hat{x} := Bb$.

2. Express the following matrix A as a product of elementary matrices; you must write the specific elementary matrices $B^{(1)}, B^{(2)}, B^{(3)}, \ldots$ such that $A = B^{(1)}B^{(2)}B^{(3)}\cdots$.

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 8 \end{bmatrix}$$

3. **a)** Give one example of a complex-valued square matrix that simultaneously has all of the following properties: It is not normal, it is diagonalizable, and it is not invertible. Prove that your matrix has all of these properties, or else prove that such a matrix does not exist.
   b) Give one example of a complex-valued square matrix that simultaneously has all of the following properties: It is normal, it is not diagonalizable, and it is invertible. Prove that your matrix has all of these properties, or else prove that such a matrix does not exist.

4. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and you are able to evaluate $T(y)$ for any $y \in \mathbb{R}^n$. Give an explicit way to construct a matrix $A \in \mathbb{R}^{m\times n}$ such that, for all $x \in \mathbb{R}^n$, it will hold that $Ax = T(x)$. Prove that, indeed, for the A that you construct, it will hold that $Ax = T(x)$ for all $x \in \mathbb{R}^n$.

5. Let $A \in \mathbb{C}^{m\times n}$. Prove that a left inverse L (i.e. $LA = I_n$) exists if and only if $\text{rank}\, A = n$, and when this holds, give one such left inverse.

6. Let $T_n \in \mathbb{R}^{n\times n}$ be the tridiagonal matrix with 2 on the main diagonal and -1 on the sub- and super-diagonals. Let $b \in \mathbb{R}^n$ be the vector defined by $b_i = i$ $(1 \le i \le n)$.

   (a) Write the recurrence satisfied by the solution of $T_n x = b$ and give its characteristic equation.

   (b) Based on (a), solve $T_n x = b$ in closed form.