Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Spring Semester
Real Analysis

Monday, January 13, 2025

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$f_n(x) = \frac{n}{2} \int_{t=x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t,$$

for $n = 1, 2, 3, \ldots$.

(i) Show that f_n converges pointwise to some limit function and identify the function.

(ii) Prove that the convergence is uniform in any compact interval.

2. What is $\lim_{x\to\infty} \left(\frac{e^x}{1+e^x}\right)^x$?

3. Suppose X and Y are disjoint sets and $f : X \to Y$ and $g : Y \to X$ are injective (i.e., one-to-one) functions. The results of this exercise can be viewed partial building blocks for using f and g to construct a bijection between X and Y. Starting with $y_1 \in Y$ we consider infinite sequences of the form $y_1, x_1, y_2, x_2, \ldots$, with $f(x_i) = y_{i+1}$ for $i = 1, 2, \ldots$, and $g(y_i) = x_i$ for $i = 1, 2, \ldots$, which can be pictured using the following diagram:

$$y_1 \xrightarrow{g} x_1 \xrightarrow{f} y_2 \xrightarrow{g} x_2 \xrightarrow{f} y_3 \xrightarrow{g} \cdots .$$

Such a sequence will be called *initialized* if $y_1 \notin \mathrm{Im} f$, i.e., if there is no $x \in X$ such that $f(x) = y_1$.

(i) Show that an initialized sequence cannot contain repeated values, or equivalently that $x_i \neq x_j$ and $y_i \neq y_j$ whenever $i \neq j$.

(ii) Show that if $y_1, x_1, y_2, x_2, \ldots$ and $y_1', x_1', y_2', x_2', \ldots$ are initialized sequences with $y_1 \neq y_1'$, then

$$\{y_i : i = 1, 2, \ldots\} \cap \{y_i' : i = 1, 2, \ldots\} = \emptyset, \qquad \text{(a)}$$

and

$$\{x_i : i = 1, 2, \ldots\} \cap \{x_i' : i = 1, 2, \ldots\} = \emptyset. \qquad \text{(b)}$$

You do not have to show this, but as a consequence of (i) and (ii), if every element of X is an element in some initialized sequence then we can define a bijection $h : X \to Y$ as follows. For $x \in X$, by (ii) there can be at most one initialized sequence containing x and by (i) the position of x in that sequence is unambiguous, and the map h sending x to its immediate predecessor y in that sequence is well-defined. The image of this map one-to-one since by (i) and (ii) y cannot appear in more than one initialized sequence and in that sequence it can appear at most once. For any $y \in Y$, y has some $x = g(y)$ as a successor, so $y = h(x)$ and we see that h is a surjection.

4. Suppose $(x^{(n)})_{n=1}^{\infty}$ is a sequence of points in $\mathbb{R}^d$ with the property that

$$\|x^{(n+1)} - x^{(n)}\| \leq c\|x^{(n)} - x^{(n-1)}\|$$

for $n = 2, 3, \ldots$, and for some constant $c \in (0, 1)$. (Here $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$.) Show that the sequence converges to some finite limit.

5. Suppose $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and a sequence of functions $f_n : [0, +\infty) \to \mathbb{R}$ for $n = 1, 2, \ldots$ is defined by $f_n(x) := f(x^n)$. Suppose that the functions $f_1, f_2, \ldots$ are equicontinous at 1, i.e., that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f_n(x) - f_n(1)| < \epsilon \text{ for all } n = 1, 2, \ldots \text{ and } x \in (1 - \delta, 1 + \delta).$$

Show that f must be a constant function.

6. Consider the power series

$$f(x) = \sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + \cdots .$$

(i) What is the radius of convergence? Hint: Compare with other series with known radius of convergence.

(ii) Show that

$$f(x) \leq \frac{x^2}{1 - x} \text{ for all } 0 \leq x < 1.$$

Department of Applied Mathematics and Statistics
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Probability

Tuesday, January 14, 2025

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2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Suppose $n > 1$. We put n balls into n boxes independently of one another so that each ball is equally likely to be put into any of the n boxes. Compute the probability that box number 1 is the only empty box.

2. For each integer $n \geq 1$, X_n is discrete uniform on the integers from 1 through n inclusive, and let N be geometrically distributed with probability of success $0 < p < 1$ independent of all the X_n's. Compute $E(X_N)$ and simplify as much as possible. Recall: $P(N = n) = p(1-p)^{n-1}$ for $n = 1, 2, 3, \ldots$; and, $P(X_n = j) = \frac{1}{n}$ for $j = 1, 2, \ldots, n$.

3. Suppose $n > 1$. We flip a fair coin until the n-th head occurs and we stop. If this nth head occurs on the $2n$-th flip of the coin, what is the probability that the $(n-1)$-st head happens on flip $2n-1$? Simplify completely.

4. Let X and Y be independent and identically distributed unit exponential random variables, i.e., they each have the PDF $f(x) = e^{-x}$ for $x > 0$. Compute the conditional probability that $X < 2$ given that $X/Y > 1$.

5. Let $a > 0$ be a fixed positive constant, and suppose that in a crowd of size n the probability that a particular person has a trait is a/n independent from person to person. If it's known that as n tends to ∞, the probability the crowd has nobody with the trait is $1/e^2$. Determine with justification the value of a.

6. Consider rectangles whose base and height are independent and uniformly distributed on the interval $[0, 1]$. Find the PDF of the area of the rectangle.

Department of Applied Mathematics and Statistics
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Introductory Examination–Spring Semester
Linear Algebra

Wednesday, January 15, 2025

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $S \in \mathbb{R}^{n \times n}$ be a symmetric and invertible matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Consider a polynomial $p(x)$ of degree at most $n-1$:

$$p(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_{n-1} x^{n-1}.$$

Suppose you are given values $y_1, y_2, \ldots, y_n \in \mathbb{R}$ and you wish to determine coefficients $h_0, h_1, \ldots, h_{n-1}$ such that $p(\lambda_i) = y_i$ for all $i = 1, 2, \ldots, n$.

   (a) Write the linear system corresponding to this problem, i.e., write the problem in the form $Ab = c$, specifying the $n \times n$ matrix A and the column vectors b and c.

   (b) What conditions must S (or equivalently, its eigenvalues $\lambda_1, \ldots, \lambda_n$) satisfy for the system in (a) to have a unique solution? Explain.

2. Verify that given an invertible matrix $A \in \mathbb{R}^{n \times n}$ and $u, v \in \mathbb{R}^n$ such that $1 + v^T A^{-1} u \neq 0$, then

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}.$$

3. Let $A, B \in \mathbb{R}^{n \times n}$. Suppose that (i) AB is symmetric and that (ii) A and B commute. This problem has two parts:

   (a) Prove that A does not need to be symmetric by providing a counterexample, i.e., give an example of matrices A and B satisfying conditions (i) and (ii) for which A is not symmetric.

   (b) Give conditions on B such that for all A satisfying (i) and (ii) we can conclude that A is symmetric.

   Include a clear proof and explanation for both parts.

4. Let $U, V \in \mathbb{C}^{n \times n}$. Assume that $UV = VU$ and that U is Hermitian. Prove that there exists a basis in which U is diagonal and V is block diagonal, with each block corresponding to an eigenspace of U.

5. Consider the following matrix:

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^{u \times u}$, $B \in \mathbb{R}^{u \times w}$, $C \in \mathbb{R}^{w \times u}$, and $D \in \mathbb{R}^{w \times w}$.

Assuming A is invertible, define $S = D - CA^{-1}B$.

(a) Prove that T is invertible if and only if S is also invertible.

(b) Assuming invertible S, prove $\text{rank}(T) = \text{rank}(A) + \text{rank}(S)$.

Hint: Start from the factorization:

$$T = \begin{bmatrix} A & 0 \\ C & S \end{bmatrix} \begin{bmatrix} I_u & A^{-1}B \\ 0 & I_w \end{bmatrix}.$$

6. Let $A \in \mathbb{R}^{m \times n}$ with singular value decomposition:

$$A = U \Sigma V^T$$

where Σ is a rectangular matrix with diagonal entries $\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0$.

Let P_R denote the orthogonal projection matrix onto the range of A.

(a) Write P_R as a function of U.

(b) Prove $P_R = AA^\dagger$, where $A^\dagger$ is the Moore-Penrose pseudoinverse.