Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Tuesday, January 21, 2014

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A vector or matrix is referred to as *integral* if all of its entries are integers. Let A be a nonsingular $n \times n$ integral matrix. Show $A^{-1}b$ is integral for all integral column n-vectors b if and only if $\det(A) = 1$ or -1.

Solution: If $\det(A) = 1$ or -1, then the result follows from Cramer's rule, or the cofactor formula for the inverse: $A^{-1} = \frac{C^T}{\det(A)}$; Since C^T also has integer entries and $\det(A) = 1$ or -1, A^{-1} has integer entries and so $A^{-1}b$ is an integer vector.

Conversely, if $A^{-1}b$ is integral for every integral b, $A^{-1}e^i$ is integral, where e^i is the i-th standard unit vector. Thus, every column of A^{-1} has integer entries. Therefore, $\det(A^{-1})$ is an integer. However, $\det(A)\det(A^{-1}) = \det(I) = 1$ and both $\det(A)$ and $\det(A^{-1})$ are integers. Thus, they have to be 1 or -1.

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2. For any nonnegative integer-valued random variable, define its probability generating function by $\Phi_X(t) = E(t^X)$ for all real t.

(a) For any $a > 0$, show that for $0 \le t \le 1$

$$P(X \le a) \le \frac{\Phi_X(t)}{t^a}.$$

(b) Use the above result to show for a Poisson random variable N with parameter λ and for any $a \in [0, \lambda]$ that

$$P(N \le a) \le e^{-\lambda} \left(\frac{e\lambda}{a}\right)^a$$

Solution: (a) Clearly, $X \le a \iff t^X \ge t^a$ for $0 \le t \le 1$. Thus,

$$P(X \le a) = P(t^X \ge t^a) \le E(\frac{t^X}{t^a} : t^X \ge t^a) \le \frac{E(t^X)}{t^a}.$$

(b) For a Poisson random variable N

$$\Phi_N(t) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} = e^{\lambda(t-1)}.$$

Thus, the bound in (a) becomes

$$P(N \le a) \le \frac{e^{\lambda(t-1)}}{t^a} = e^{\lambda(t-1) - a\ln t}$$

for all $t \in [0,1]$. Minimizing the exponent over t one finds that $t = a/\lambda$. Furthermore, $t \in [0,1]$ when $a \in [0,\lambda]$. Thus, the sharpest bound obtained for $t = a/\lambda$ is

$$P(N \leq a) \leq \frac{e^{a-\lambda}}{(a/\lambda)^a} = e^{-\lambda}\left(\frac{e\lambda}{a}\right)^a.$$

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3. Assuming that temperature varies continuously with location, prove that there are, at any given time, antipodal points on the equator of the earth that have the same temperature.

Solution: By considering the temperature along any great circle, the problem translates into the following math problem. Let $f : [0, 2\pi] \to \mathbb{R}$ be continuous, and suppose $f(0) = f(2\pi)$. Prove that there exists $c \in [0, \pi]$ such that $f(c) = f(c + \pi)$.

For this, consider $g : [0, \pi] \to \mathbb{R}$ defined by $g(x) := f(x + \pi) - f(x)$. Then g is continuous, with $g(0) = f(\pi) - f(0) = f(\pi) - f(2\pi) = -g(\pi)$. If $g(0) = 0$, we can take $c = 0$. Otherwise, $g(0)$ and $g(\pi)$ have opposite sign and so, by the intermediate value theorem, $g(c) = 0$ (as desired) for some $c \in [0, \pi]$.

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4. Suppose the probability that a family will have n children is 2^{-n-1} and that each child is equally likely to be male or female, independently of the other children. What is the conditional probability that a family has at least one child given it has no boys?

Solution: Let N be the number of children and let B be the number of boys.

$$\begin{aligned} P(B=0) &= \sum_{n=0}^{\infty} P(B=0, N=n) \\ &= \sum_{n=0}^{\infty} P(B=0|N=n)P(N=n) \\ &= \sum_{n=0}^{\infty} 2^{-n}2^{-n-1} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} 4^{-n} \\ &= \frac{2}{3} \end{aligned}$$

Now $P(B = 0, N \geq 1) = P(B = 0) - P(B = 0, N = 0) = P(B = 0) - \frac{1}{2} = \frac{1}{6}$ since $P(B = 0, N = 0) = P(N = 0)$. Hence, $P(N \geq 1|B = 0) = \frac{1}{6}/\frac{2}{3} = \frac{1}{4}$.

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5. Suppose the probability that the Dow Jones Stock Index increases today is 0.54, that it increases tomorrow is also 0.54, and that it rises on both days is 0.28. Find (with explanation) the probability that it increases on *neither* day.

Solution: Let D be the event that the Index rises today, and M the event that it rises tomorrow. Then the desired probability is

$$\begin{aligned} P((D \cup M)^c) &= 1 - P(D \cup M) = 1 - [P(D) + P(M) - P(D \cap M)] \\ &= 1 - [0.54 + 0.54 - 0.28] = 1 - 0.80 = 0.20. \end{aligned}$$

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6. Let $A_m \in \mathbf{R}^{4\times 4}$, $m = 1, 2, \cdots$, defined by

$$A_m = \begin{bmatrix} 1 & 2^m & 3^m & 4^m \\ 3^m & 1 & 4^m & 2^m \\ 2^m & 4^m & 1 & 3^m \\ 4^m & 3^m & 2^m & 1 \end{bmatrix}$$

Determine for which values of m the matrix A_m is invertible and justify your answer.

Solution: Let

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} 1 & 2^m \\ 3^m & 1 \end{bmatrix}, \quad C_m = \begin{bmatrix} 2^m & 4^m \\ 4^m & 3^m \end{bmatrix}$$

Then A_m can be written as

$$A_m = \begin{bmatrix} B_m & JC_mJ \\ C_m & JB_mJ \end{bmatrix}$$

and it is similar to

$$D_m := \begin{bmatrix} B_m - JC_m & 0 \\ 0 & B_m + JC_m \end{bmatrix}$$

via an orthogonal similarity matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -J \\ I & J \end{bmatrix}$$

Since D_m is block diagonal, we have $\det(A_m) = \det(B_m - JC_m)\det(B_m + JC_m)$. Since

$$\det(B_m - JC_m) = (1 - 4^m)^2 - (2^m - 3^m)(3^m - 2^m) = (4^m - 1)^2 + (3^m - 2^m)^2 > 0$$

for all $m \geq 1$, we only need to investigate $\det(B_m + JC_m)$. We note that

$$\det(B_m + JC_m) = (1 + 4^m)^2 - (2^m + 3^m)^2 =: f_m,$$

and $f_1 = 0$, $f_2 = 17^2 - 13^2 > 0$. Since it can be shown that the sequence f_m is an increasing sequence on m, we conclude that $\det(B_m + JC_m) > 0$, hence $\det(A_m) > 0$ for all $m \geq 2$. For $m = 1$, $\det(A_1) = 0$ from $\det(B_1 + JC_1) = f_1 = 0$. Therefore A_m is invertible for all $m \geq 2$ but it is not for $m = 1$.

Alternative solution: If $m = 1$ then A_m is singular since the sum of the first and fourth columns equals the sum of the second and third columns.

For $m > 1$ observe that by permuting columns of A_m we obtain

$$\begin{bmatrix} 4^m & 3^m & 2^m & 1 \\ 2^m & 4^m & 1^m & 3^m \\ 3^m & 1 & 4^m & 2^m \\ 1 & 2^m & 3^m & 4^m \end{bmatrix}$$

so proving that A_m is invertible amounts to proving invertibility of this matrix. Clearly, for $m = 2$ we have

$$1 + 2^m + 3^m < 4^m.$$

Assuming this inequality holds for some $m > 1$ we have

$$1 + 2^{m+1} + 3^{m+1} < (1 + 2^m + 3^m) \times 3 < 4^m \times 4 = 4^{m+1}.$$

So for $m > 1$ the matrix is *diagonally dominant* i.e. the sum of the absolute values of the non-diagonal entries in a given row (column) is less than the absolute value of the diagonal entry in that row (column). Invertibility follows from the Gershgorin circle theorem.

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7. Consider the one-parameter family of second-order homogeneous linear differential equations of the form

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + [\lambda\rho(x) - q(x)]u = 0, \quad a \leq x \leq b \tag{1}$$

with *endpoint* (or boundary) *conditions* $u(a) = u(b) = 0$, where λ is a parameter, the functions p, ρ, q are continuous on $[a, b]$, and p, ρ are positive on $[a, b]$. Let $u(x)$ and

$v(x)$ be solutions of (**??**) corresponding to distinct parameters λ and μ, respectively, that satisfy the given endpoint conditions. Show that

$$\int_a^b \rho(x)u(x)v(x)dx = 0$$

Solution: By hypothesis, we have the following equations :

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + [\lambda\rho(x) - q(x)]u = 0, \quad a \leq x \leq b \tag{2}$$

and

$$\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + [\mu\rho(x) - q(x)]v = 0, \quad a \leq x \leq b \tag{3}$$

Multiply (**??**) by $v(x)$ and integrate both sides from a to b to obtain:

$$\int_a^b v(x)\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]dx + \int_a^b \lambda\rho(x)u(x)v(x)dx - \int_a^b q(x)u(x)v(x)dx = 0$$

Using integration by parts on the first term above, we obtain :

$$\int_a^b \frac{d}{dx}\left[p\left(x\right)\frac{du}{dx}\right]v\left(x\right)dx = \left[p\left(x\right)\frac{du}{dx}v\left(x\right)\right]_a^b - \int_a^b p\left(x\right)\frac{du}{dx}\frac{dv}{dx}dx$$

The first term on the right hand side is 0 because of the boundary conditions.

Thus, we obtain

$$-\int_a^b p\left(x\right)\frac{du}{dx}\frac{dv}{dx}dx + \int_a^b \lambda\rho(x)u(x)v(x)dx - \int_a^b q(x)u(x)v(x)dx = 0 \tag{4}$$

Similarly, by multiplying (**??**) by $u(x)$ and integrating both sides from a to b, we obtain

$$-\int_a^b p\left(x\right)\frac{du}{dx}\frac{dv}{dx}dx + \int_a^b \mu\rho(x)u(x)v(x)dx - \int_a^b q(x)u(x)v(x)dx = 0 \tag{5}$$

Subtracting (**??**) from (**??**), we get

$$\int_a^b (\lambda - \mu)\rho(x)u(x)v(x)dx = 0$$

Since $\lambda \neq \mu$, this implies that $\int_a^b \rho(x)u(x)v(x)dx = 0$

8. Let (x_n) be a sequence of positive numbers, and denote the average of the first n entries by

$$\bar{x}_n = (x_1 + \cdots + x_n)/n.$$

Let $N = (n_k)$ be a subsequence of $\mathbb{N}$ with $\lim_{k\to\infty} n_{k+1}/n_k = r > 0$. Prove: if the sequence $(\bar{x}_n)$ converges along N to x, then

$$x/r \le \liminf \bar{x}_n \le \limsup \bar{x}_n \le r \cdot x \ .$$

Solution: For $n_k \le n < n_{k+1}$, the positivity of the x_n yields

$$\frac{n_k}{n_{k+1}} \bar{x}_{n_k} \le \bar{x}_n \le \bar{x}_{n_{k+1}} \cdot \frac{n_{k+1}}{n_k} \ .$$

As $n \to \infty$, the integer k for which $n_k \le n < n_{k+1}$ tends to $+\infty$, and our assumptions imply that the left-most member converges to x/r and the right-most to rx.

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9. If x is a real number, we use $\lfloor x \rfloor$ to denote its floor, that is, the largest integer that is less than or equal to x. Show that $\lim_{n\to\infty} n!e - \lfloor n!e \rfloor = 0$.

Solution: Note that $x \ge \lfloor x \rfloor$ for all x. We can write $e = \sum_{k=0}^{\infty} 1/k!$ so that $n!e = M_n + F_n$ where $M_n = n! \sum_{k=0}^{n} 1/k!$, which is clearly a positive integer, and $F_n = n! \sum_{k=n+1}^{\infty} 1/k!$. Now

$$F_n = 1/(n+1) + 1/((n+1)(n+2)) + 1/((n+1)(n+2)(n+3)) + \cdots =$$

$$\{1 + 1/(n+2) + 1/((n+2)(n+3)) + 1/((n+2)(n+3)(n+4)) + \cdots\}/(n+1)$$
$$\le \left\{1 + 1/(n+1) + 1/(n+1)^2 + 1/(n+1)^3 + \cdots \right\}/(n+1)$$
$$= 1/((1 - 1/(n+1))(n+1)) = 1/n < 1.$$

So we conclude that that

$$\lfloor n!e \rfloor = M_n,$$

and

$$n!e - \lfloor n!e \rfloor = F_n \le 1/n,$$

and the result follows.

10. Suppose A and B are $n \times n$ matrices such that $I - AB$ is invertible with inverse X. Show that $I - BA$ is invertible and find a simple expression for its inverse.

Hint: To arrive at a good guess for the inverse, write down formal power series expansions for $(I - BA)^{-1}$ and $(I - AB)^{-1}$ to get a candidate for the desired inverse. Then prove that the inverse is correct without using formal power series.

Solution: Formally,

$$X := (I - AB)^{-1} = I + AB + (AB)^2 + (AB)^3 \cdots$$

and we can write

$$(I - BA)^{-1} = I + BA + (BA)^2 + (BA)^3 + \cdots = I + B(I + AB + (AB)^2 + \cdots)A = I + B(I - AB)^{-1}A.$$

So it appears that our candidate for the desired inverse is $I + BXA$ where $X = (I - AB)^{-1}$. Now we proceed to show that this works without the series expansion.

$$(I - BA)(I + BXA) = I - BA - BABXA + BXA = I - BA + B(I - AB)XA$$

$$= I - BA + BIA = I - BA + BA = I.$$

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11. Let n be a positive integer. We create a random n-digit (decimal) number as follows:

First we pick a random number k from $\{0, 1, 2, \ldots, n\}$ uniformly; that is, the probability of picking each k is $1/(n + 1)$.

Then we pick uniformly at random an n-digit number in which exactly k of the digits are 1s and the remaining $n - k$ digits are 9s. That is, given k, all such n-digit numbers are equally likely to be picked.

Call the resulting value X.

Question: What is the expected value of X? Justify your answer.

For example, suppose $n = 5$. If $k = 2$ (with probability $1/6$) then the ten numbers 99911, 99191, 99119, 91991, 91919, 91199, 19991, 19919, 19199, 11999 are equally likely (with probability $\frac{1}{6} \cdot \frac{1}{10} = \frac{1}{60}$). However, X takes the value 11111 with probability $\frac{1}{6}$. So not all values of X have the same probability.

Solution: Let X_i be the value of the i'th digit in X. Observe that it is equally likely that this digit is a 1 or a 9, i.e. $\Pr\{X_i = 1\} = \Pr\{X_i = 9\} = \frac{1}{2}$. Note that $E(X_i) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 9 = \frac{1}{2} \cdot 10 = 5$.

We have

$$X = X_0 + 10X_1 + 10^2 X_2 + \cdots + 10^{n-1} X_n$$

and so by linearity of expectation

$$\begin{aligned} E(X) &= E(X_0) + 10E(X_1) + 10^2 E(X_2) + \cdots + 10^{n-1} E(X_n) \\ &= 5\left[1 + 10 + 100 + \cdots + 10^{n-1}\right] \\ &= 5\left(\frac{10^n - 1}{10 - 1}\right) \end{aligned}$$

or written in decimal

$$E(X) = \underbrace{5555\ldots5}_{n \text{ digits}}.$$

Alternative solution: The distribution of X, which is an n digit number whose digits are all 1's and 9's has the following symmetry property. For every n digit number x all of whose digits are 1's and 9's, if $\tilde{x}$ is the number obtained by *flipping* the digits of x (changing 1's to 9's and 9's to 1's) then $P[X = x] = P[X = \tilde{x}]$. For each pair of such values $x, \tilde{x}$ let p denote the ordered pair $[x, \tilde{x}]$ where $x < \tilde{x}$. Observe $\frac{x+\tilde{x}}{2} = \underbrace{5555\ldots5}_{n \text{ digits}}$ for all pairs p. We can replace the sum defining the expected value of X by a sum over pairs to obtain

$$E[X] = \sum_x xP[X = x] = \sum_p xP[X = x] + \tilde{x}P[X = \tilde{x}] = \sum_p (x + \tilde{x})P[X = x] =$$

$$\sum_p \underbrace{5555\ldots5}_{n \text{ digits}} \times 2P[X = x] = \underbrace{5555\ldots5}_{n \text{ digits}} \sum_p P[X = x] + P[X = \tilde{x}] = \underbrace{5555\ldots5}_{n \text{ digits}}.$$

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12. For which real value(s) of x does the following series converge and diverge:

$$\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right).$$

Justify your answer.

Solution: This series converges absolutely for all real x. Here's why:

For $y \geq 0$,

$$|\sin(y)| = \left|\int_0^y \cos(u)\, du\right| \leq \int_0^y |\cos(u)|\, du \leq \int_0^y 1\, du = y.$$

If $y < 0$, then $-y > 0$ and $|\sin(y)| = |-\sin(-y)| \leq -y$. Therefore, for all real y, $|\sin(y)| \leq |y|$. Now,

$$\left|\sum_{n=1}^{\infty} \sin(\frac{x}{n^2})\right| \leq \sum_{n=1}^{\infty} |\sin(\frac{x}{n^2})| \leq |x| \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

for all real x.

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13. Consider the vector space $V = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \text{ real}, n \text{ fixed}\}$ of all polynomials of degree less than or equal to n and the *derivative transformation* $D : V \to V$ sending polynomials in V to their derivatives. Determine the following: the matrix of D with respect to the basis $1, x, x^2, \ldots, x^n$ of V; the rank of $D : V \to V$; and the nullspace of D.

Solution: Clearly, $D(1) = 0$ and for any $1 \leq k \leq n$, $D(x^k) = kx^{k-1}$. Therefore,

$$D = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & n-1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Since this $(n+1) \times (n+1)$ matrix is in echelon form the rank of D is the number of pivot entries, which is n. The nullspace of D is the subspace of polynomials in V that get mapped to the zero polynomial under D, i.e., the constant polynomials: $\text{nullspace}(D) = \{a_0 : a_0 \text{ real}\}$.

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14. Cards are drawn one by one, at random, and without replacement from a standard deck of 52 playing cards. What is the probability that the fourth Heart is drawn on the tenth draw? (Do not simplify to a decimal number.)

Solution: Let A be the event that the tenth draw is a Heart, and B be the event that there are exactly three Hearts drawn in the first nine draws.

Rephrasing its description, A is the event that in choosing 9 cards, 3 are chosen from the 13 Hearts and 6 from the 39 non-Hearts, so using an equally likely probability model, we have

$$P[A] = \frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}}.$$

Conditional upon event A, B is the event that a randomly chosen card drawn from a (partial) deck of 10 Hearts and 33 non-Hearts is a Heart. Thus,

$$P[B|A] = \frac{10}{43}.$$

By the multiplication rule for the probability of an intersection event,

$$P[AB] = P[B|A]P[A] = \frac{10}{43}\frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}}.$$

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15. Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric and positive semidefinite matrices. Show that

$$tr(AB) \geq 0,$$

where tr denotes the trace, i.e., the sum of the entries on the diagonal.

Solution: Since B is positive semi-definite, we can decompose

$$B = \mu_1 b_1 b_1^T + \mu_2 b_2 b_2^T + \ldots + \mu_n b_n b_n^T$$

where $\mu_1, \ldots, \mu_n$ are the (nonnegative) eigenvalues of B, and $b_1, \ldots, b_n$ are the eigenvectors of B. Thus,

$$\begin{aligned} tr(AB) &= tr(A(\textstyle\sum_{i=1}^n \mu_i b_i b_i^T)) \\ &= tr(\textstyle\sum_{i=1}^n (\mu_i A b_i b_i^T)) \\ &= \textstyle\sum_{i=1}^n tr(\mu_i A b_i b_i^T) \\ &= \textstyle\sum_{i=1}^n \mu_i tr(A b_i b_i^T) \\ &= \textstyle\sum_{i=1}^n \mu_i (b_i^T A b_i) \end{aligned}$$

where in the last equality, we use the observation that $tr(Ab_i b_i^T) = b_i^T A b_i$.

Since A is positive semidefinite, $b_i^T A b_i \geq 0$ for each $i = 1, \ldots, n$. Since each $\mu_i \geq 0$, the last sum above is a nonnegative real number, showing that $tr(AB) \geq 0$.

Alternative solution: Recall that any real symmetric positive semidefinite matrix C can be written as $C^{1/2}C^{1/2}$ for some real symmetric square matrix $C^{1/2}$. By factoring A and B in this manner we can write

$$tr(AB) = tr(A^{1/2}A^{1/2}B^{1/2}B^{1/2}) = tr(B^{1/2}A^{1/2}A^{1/2}B^{1/2})$$

Observe that if we define $C = B^{1/2}A^{1/2}$ then by symmetry of the matrices $A^{1/2}$ and $B^{1/2}$ we have $C^t = (A^{1/2})^t(B^{1/2})^t = A^{1/2}B^{1/2}$ and hence $tr(AB) = tr(CC^t)$. Since CC^t is symmetric $((CC^t)^t = (C^t)^t C^t = CC^t)$ and positive semidefinite $(v^t(CC^t)v = (Cv)^t(Cv) \geq 0$ its trace is the sum of its eigenvalues, all of which are nonnegative.
