

Department of Applied Mathematics and Statistics  
The Johns Hopkins University

INTRODUCTORY EXAMINATION—WINTER SESSION  
MORNING EXAM—REAL ANALYSIS

Tuesday, January 17, 2023

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is  $2/3$  of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
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4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
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7. **No calculators of any sort are needed or permitted.**

- Let the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^n + e^{cx}$  where  $n$  is an odd positive integer and  $c > 0$ .
  - Prove that  $f(x) = 0$  for some  $x \in \mathbb{R}$ .
  - Prove that the solution to  $f(x) = 0$  is unique.

*Solution:*

- Since  $\lim_{x \rightarrow -\infty} f(x) = -\infty$  and  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $f$  is continuous, the existence of a root is a consequence of the intermediate value theorem.
  - $f$  is strictly increasing so uniqueness is immediate.
- 

- Define a sequence

$$x_n = \left( \sum_{k=1}^n \frac{1}{k} \right) - \log n$$

for  $n = 1, 2, \dots$ . Prove that  $\lim_{n \rightarrow \infty} x_n$  exists and is finite.

Hint: Show that the sequence  $(x_n)$  is nonnegative by comparing the sum in  $x_n$  to a sum of integrals. Then show that the sequence  $(x_n)$  is monotone.

*Solution:* We have

$$\log n = \int_{t=1}^n \frac{1}{t} dt = \sum_{k=1}^{n-1} \int_{t=k}^{k+1} \frac{1}{t} dt.$$

so

$$x_n = \left( \sum_{k=1}^{n-1} \frac{1}{k} - \int_{t=k}^{k+1} \frac{1}{t} dt \right) + \frac{1}{n}.$$

Since the integrand in  $\int_{t=k}^{k+1} \frac{1}{t} dt$  is bounded above by  $1/k$  and we are integrating over an interval of length 1, this integral is bounded above by  $1/k$  and we obtain

$$x_n \geq 1/n.$$

Now

$$x_{n+1} - x_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log \left( \frac{n+1}{n} \right)$$

$$= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) = \frac{1}{n+1} - \int_{t=1}^{1+\frac{1}{n}} \frac{1}{t} dt.$$

The integrand is bounded below by  $\frac{1}{1+\frac{1}{n}}$  and the length of the interval of integration is  $1/n$  so

$$x_{n+1} - x_n \leq \frac{1}{n+1} - \frac{1}{n} \frac{1}{1+\frac{1}{n}} = 0.$$

Since the sequence is bounded below by zero and monotone nonincreasing, it must converge.

3. (a) Show there exists  $\epsilon > 0$  such that  $\log(1-x) \geq -2x$  for  $x \in [0, \epsilon)$ .  
 (b) Define  $p_n = \prod_{i=1}^n \frac{2^i-1}{2^i}$  for  $n = 1, 2, \dots$ . Use the result in (a) to show that the limit  $\lim_{n \rightarrow \infty} p_n$  exists and is positive.

*Solution:* (a) Observe that the inequality holds for  $x = 0$ . Define  $f(x) = \frac{\log(1-x)}{x}$  for  $x \in (0, 1)$ . Using L'hôpital's rule we see that

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} -\frac{1}{1-x} = -1$$

so we can define  $f(0) = -1$  and  $f$  is continuous in  $[0, 1)$ . Since  $f(0) = -1 > -2$ , by continuity there exists  $\epsilon > 0$  such that  $f(x) > -2$  for  $x \in [0, \epsilon)$  and we conclude that  $\log(1-x) \geq -2x$  for  $x \in [0, \epsilon)$ .

Alternative solutions to (a): One can employ the mean value theorem to the function  $g(x) := \log(1-x)$ . Since  $g'(x) = -1/(1-x)$  for  $x \in [0, 1)$ , for each such  $x$  we have

$$\frac{\log(1-x)}{x} = \frac{\log(1-x) - 0}{x - 0} = -\frac{1}{1-\xi}$$

for some  $\xi \in (0, x)$ . If  $x \in (0, 1/2]$ , then  $\xi \in (0, 1/2)$ , and so

$$\frac{\log(1-x)}{x} = -\frac{1}{1-\xi} > -2.$$

So with  $\epsilon := 1/2$  we have the desired inequality for  $x \in (0, \epsilon)$ , and the desired inequality holds with equality when  $x = 0$ .

Here's an even easier solution. Let  $h(x) := \log(1-x) + 2x$  for  $x \in [0, 1)$ . Then  $h(0) = 0$  and  $h'(x) = 2 - (1-x)^{-1}$ , which is nonnegative for  $x \in [0, 1/2]$ . Hence  $h(x) \geq h(0) = 0$  for  $x \in [0, 1/2]$ .

(b) Note that the terms in  $\log p_n = \sum_{i=1}^n \log(1 - \frac{1}{2^i})$  are all negative, so  $\log p_n$  is monotone decreasing and hence converges to some limit (possibly  $-\infty$ ). It suffices therefore to find a finite lower bound for the sum.

From (a), let  $\epsilon > 0$  be such that

$$\log(1 - x) \geq -2x$$

for  $x \in [0, \epsilon)$ . Take  $I$  such that  $1/2^I < \epsilon$ . Then for  $n \geq I$  we have

$$\begin{aligned} \log p_n &= \sum_{i=1}^n \log \left( 1 - \frac{1}{2^i} \right) = \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) + \sum_{i=I}^n \log \left( 1 - \frac{1}{2^i} \right) \\ &\geq \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2 \sum_{i=I}^n \frac{1}{2^i} \\ &\geq \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2 \sum_{i=I}^{\infty} \frac{1}{2^i} \\ &= \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2^{2-I}. \end{aligned}$$

4. A sequence of functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  for  $n = 1, 2, \dots$  is said to be *uniformly equicontinuous* if for any  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|f_n(x) - f_n(y)| < \epsilon$  for all  $n$  whenever  $|x - y| < \delta$ .

Suppose the sequence of functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  for  $n = 1, 2, \dots$  is uniformly equicontinuous, and suppose the pointwise limit  $\lim_{n \rightarrow \infty} f_n(x)$  exists for all  $x \in \mathbb{R}$ . Show that the function defined by  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  is continuous.

*Solution:* Fix  $\epsilon > 0$ . By uniform equicontinuity, there exists  $\delta > 0$  such that  $|f_n(x) - f_n(y)| < \epsilon/3$  for all  $n$  whenever  $|x - y| < \delta$ . Fix  $x, y$  with  $|x - y| < \delta$ . Since  $f_n(x) \rightarrow f(x)$  there exists  $N_1$  such that  $|f_n(x) - f(x)| < \epsilon/3$  for all  $n \geq N_1$ , and since  $f_n(y) \rightarrow f(y)$  there exists  $N_2$  such that  $|f_n(y) - f(y)| < \epsilon/3$  for all  $n \geq N_2$ . For  $n \geq \max\{N_1, N_2\}$  the triangle inequality gives

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

5. Suppose  $U \subset \mathbb{R}$  is open. Show that

$$U \subseteq \overline{U \cap \mathbb{Q}}.$$

Here  $\overline{V}$  denotes the closure of a set  $V \subseteq \mathbb{R}$ , and  $\mathbb{Q}$  denotes the set of rational numbers.

*Solution:* Given  $u \in U$ , it suffices to show every open neighborhood  $V$  of  $u$  contains a point in  $U \cap \mathbb{Q}$ . Let  $V$  be an open neighborhood of  $u$ . Then  $V \cap U$  is also an open neighborhood of  $u$ , and since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  we have  $(V \cap U) \cap \mathbb{Q} \neq \emptyset$ . Thus

$$V \cap (U \cap \mathbb{Q}) = (V \cap U) \cap \mathbb{Q} \neq \emptyset.$$

Alternative solution: Fix  $u \in U$ . Since  $U$  is open, there exists  $N$  such that  $B(u, 1/N) \subseteq U$ . We can define a sequence  $(q_n)_{n=N}^{\infty} \subseteq U \cap \mathbb{Q}$  as follows. For any  $n \geq N$ , we have  $B(u, 1/n) \subseteq B(u, 1/N) \subseteq U$ . Using the fact that  $\mathbb{Q}$  is dense in  $\mathbb{R}$  and  $B(u, 1/n)$  is open, there exists  $q_n \in \mathbb{Q} \cap B(u, 1/n)$ , and as a consequence  $q_n \in U$ . Since  $|q_n - u| < 1/n$  we conclude that  $\lim_{n \rightarrow \infty} q_n = u$ .

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Department of Applied Mathematics and Statistics  
The Johns Hopkins University

INTRODUCTORY EXAMINATION—SUMMER SESSION  
MORNING EXAM—PROBABILITY

Wednesday, January 18, 2022

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1. In a box there are 99 fair coins and 1 double-headed coin (i.e., it always comes up heads). Suppose that I randomly draw a coin, flip it  $n$  times, and it comes up heads every time. What is the smallest  $n$  such that, given this information, the probability that the coin is double-headed is at least  $\frac{9}{10}$ ?

*Solution:* Let  $H_n$  denote the event that the  $n$  tosses are all heads, and  $A$  denote the event that I drew the double-headed coin. By Bayes's Rule,

$$\begin{aligned} P(A|H_n) &= \frac{P(A \cap H_n)}{P(H_n)} = \frac{P(A)P(H_n|A)}{P(A)P(H_n|A) + P(A^c)P(H_n|A^c)} \\ &= \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n}. \end{aligned}$$

We solve

$$\begin{aligned} P(A|H_n) &\geq \frac{9}{10} \\ \iff \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \frac{1}{2}^n} &\geq \frac{9}{10} \\ \iff \frac{1}{100} &\geq 9 \cdot \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n \\ \iff 2^n &\geq 9 \cdot 99 \\ \iff n &\geq 10, \end{aligned}$$

keeping in mind that  $n$  must be a natural number. The smallest  $n$  is 10.

2. In a group of  $n$  people, suppose that each person has equal probability  $\frac{1}{365}$  of having their birthday on any day of the year (assume Feb 29th is not possible), and the birthdays are independent. We say a birthday is *unique* if exactly one person in the group has that birthday. What is the expected number of unique birthdays, as a function of  $n$ ?

*Solution:* Number the days of the year from 1 to 365 and the people from 1 to  $n$ . Let  $X_k$  be the indicator random variable that day  $k$  is a unique birthday, and  $U$  be the total number of unique birthdays. Note that  $U = \sum_{k=1}^{365} X_k$ . By linearity of expectation,

$$\mathbb{E}U = \sum_{k=1}^{365} \mathbb{E}X_k = \sum_{k=1}^{365} P(k \text{ is a unique birthday}).$$

Now

$$\begin{aligned}
 &P(k \text{ is a unique birthday}) \\
 &= \sum_{m=1}^n P(\text{person } m \text{ has birthday } k, \text{ and everyone else has a birthday } \neq k) \\
 &= \sum_{m=1}^n \frac{1}{365} \left(\frac{364}{365}\right)^{n-1} = \frac{n}{365} \left(\frac{364}{365}\right)^{n-1}.
 \end{aligned}$$

Thus

$$\mathbb{E}U = 365 \cdot \frac{n}{365} \left(\frac{364}{365}\right)^{n-1} = \boxed{n \left(\frac{364}{365}\right)^{n-1}}.$$

3. 6 ducks and 6 geese randomly sit down in a circle, with each permutation equally likely. Compute the probability that there are at least 4 ducks in a row somewhere in the circle.

*Solution:* Label the positions clockwise from 1 to 12, and abbreviate ducks as D and geese as G. We note that there are at least 4 ducks in a row somewhere in the circle if and only if, starting from some location  $i$  and reading clockwise, the birds at those locations are GDDDD. Letting  $A_i$  denote this event, we wish to find the probability of  $\bigcup_{i=1}^{12} A_i$ .

Note

$$P(A_i) = \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8},$$

as we may as well assume we are drawing ducks and geese randomly starting at the  $i$ th location, and are sampling the ducks and geese without replacement. The  $A_i$  are disjoint, as it is impossible to have two sequences of at least 4 ducks. Hence

$$P\left(\bigcup_{i=1}^{12} A_i\right) = \sum_{i=1}^{12} P(A_i) = 12 \cdot \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8} = \boxed{\frac{3}{11}}.$$

4. Let  $X_1$ ,  $X_2$ , and  $X_3$  be independent random variables drawn from the exponential distribution with mean 1. Compute the probability density function of  $X_2$  conditional on the event  $X_1 \leq X_2 \leq X_3$ .

(Equivalently, let  $Y_1 \leq Y_2 \leq Y_3$  be  $X_1, X_2, X_3$  sorted in increasing order. Compute the probability density function of  $Y_2$ .)



*Solution:*

**Solution 1.** The joint density function of  $(X_1, X_2, X_3)$  is  $e^{-(x_1+x_2+x_3)}$ , for  $x_1, x_2, x_3 \geq 0$ . The density function of  $X_2$  conditional on  $X_1 \leq X_2 \leq X_3$  is (for  $x_2 \geq 0$ )

$$\frac{\int_{x_1, x_3: 0 \leq x_1 \leq x_2 \leq x_3} e^{-(x_1+x_2+x_3)} dx_1 dx_3}{\int_{0 \leq x_1 \leq x_2 \leq x_3} e^{-(x_1+x_2+x_3)} dx_1 dx_2 dx_3}.$$

Let  $f(x_2)$  denote the numerator and  $Z$  denote the denominator. The numerator is

$$\begin{aligned} f(x_2) &= e^{-x_2} \int_{x_2}^{\infty} e^{-x_3} \int_0^{x_2} e^{-x_1} dx_1 dx_3 \\ &= e^{-x_2} e^{-x_2} (1 - e^{-x_2}) = e^{-2x_2} (1 - e^{-x_2}). \end{aligned}$$

The denominator is

$$Z = \int_0^{\infty} f(x_2) dx_2 = \int_0^{\infty} (e^{-2x_2} - e^{-3x_2}) dx_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The conditional probability density function is then

$$\frac{f(x)}{Z} = \boxed{6e^{-2x}(1 - e^{-x})}, \quad x \geq 0.$$

**Solution 2.** Let  $F(x)$  denote the cumulative distribution function for the exponential distribution with mean 1; note  $F(x) = 1 - e^{-x}$  for  $x \geq 0$ . By symmetry, conditional on any ordering  $X_{\pi(1)} \leq X_{\pi(2)} \leq X_{\pi(3)}$ , the distribution of  $X_{\pi(2)}$  is the same. Hence, if we let  $Y_1 \leq Y_2 \leq Y_3$  be  $X_1, X_2, X_3$  sorted in increasing order, then an equivalent problem is to find the density function of  $Y_2$ .

Note that when  $F$  is a continuous cdf, the inverse cdf  $F^{-1}$  transforms a uniform random variable to a random variable with cdf  $F$ . Because  $F^{-1}$  is monotonic,  $(Y_1, Y_2, Y_3)$  has the same distribution as  $(F^{-1}(Z_1), F^{-1}(Z_2), F^{-1}(Z_3))$ , where  $(Z_1, Z_2, Z_3)$  are uniform order statistics. We know that  $Z_2$  has a beta distribution with  $\alpha = 2$  and  $\beta = 2$ , which has density  $g(x) := \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} x^{\alpha-1} (1-x)^{\beta-1} = 6x(1-x)$  for  $0 \leq x \leq 1$ .

By the change-of-variables formula,  $F^{-1}(Z_2)$  then has density function

$$g(F(x))F'(x) = 6F(x)(1 - F(x))e^{-x} = 6e^{-2x}(1 - e^{-x}), \quad x \geq 0.$$

5. Let  $X = (X_1, \dots, X_n)$  be a random vector in  $\mathbb{R}^n$  whose entries are independently drawn from a standard normal distribution  $X_k \sim N(0, 1)$ . For  $x \in \mathbb{R}^n$ , let  $\|x\| =$

$\sqrt{x_1^2 + \cdots + x_n^2}$ . For any  $\epsilon > 0$ , show that there exists a constant  $C_\epsilon$  such that for all large enough  $n$ ,

$$P(\|X\| - \sqrt{n} > C_\epsilon) < \epsilon.$$

*Solution:*

**Solution 1.** Let  $S_n = X_1^2 + \cdots + X_n^2$ ; note  $\|X\|^2 = S_n$ . Then for any  $C > 0$ ,

$$\begin{aligned} P(\|X\| - \sqrt{n} > C) &= P(\|X\| > \sqrt{n} + C) \\ &= P(\|X\|^2 > n + 2C\sqrt{n} + C^2) \\ &\leq P(S_n > n + 2C\sqrt{n}) \\ &= P\left(\frac{S_n - n}{\sqrt{n}} > 2C\right). \end{aligned}$$

Now,  $S_n$  is a sum of iid copies of  $X_1^2$ , and  $X_1^2$  has variance  $\sigma^2 := \text{Var}(X_1^2) = \mathbb{E}X_1^4 - (\mathbb{E}X_1^2)^2 = 2 < \infty$ . (The exact value is not important.) Thus by the central limit theorem,  $\frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z$ , where  $Z \sim N(0, 1)$  is standard normal. Let  $\Phi$  be the cdf of the normal distribution and  $C_\epsilon = \frac{\sigma}{2}\Phi^{-1}(1 - \frac{\epsilon}{2})$ . Then as  $n \rightarrow \infty$ ,

$$\begin{aligned} P\left(\frac{S_n - n}{\sqrt{n}} > 2C_\epsilon\right) &= P\left(\frac{S_n - n}{\sqrt{n}} > \sigma\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) \\ &\rightarrow P_{Z \sim N(0,1)}\left(\sigma Z > \sigma\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) = \frac{\epsilon}{2}, \end{aligned}$$

using convergence in distribution. For  $n$  large enough, we will still have  $P\left(\frac{S_n - n}{\sqrt{n}} > 2C_\epsilon\right) < \epsilon$ , which gives the required bound.

**Solution 2.** Defining  $S_n$  as in Solution 1, we have that

$$\|X\| - \sqrt{n} = \sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{n}} \frac{\sqrt{n}}{\sqrt{S_n} + \sqrt{n}}.$$

As before, by the central limit theorem,  $\frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z$ ,  $Z \sim N(0, 1)$ . By the weak law of large numbers,  $\frac{S_n}{n} \rightarrow \mathbb{E}X_1^2 = 1$  in probability, so  $\frac{\sqrt{n}}{\sqrt{S_n} + \sqrt{n}} = \frac{1}{\sqrt{S_n/n} + 1} \rightarrow \frac{1}{2}$  in probability. Hence  $\sqrt{S_n} - \sqrt{n} \Rightarrow \frac{1}{2}\sigma Z$ . Setting  $C_\epsilon = \frac{\sigma}{2}\Phi^{-1}(1 - \frac{\epsilon}{2})$  we have as  $n \rightarrow \infty$ ,

$$P(\sqrt{S_n} - \sqrt{n} > C_\epsilon) \rightarrow P_{Z \sim N(0,1)}\left(\frac{\sigma}{2}Z > \frac{\sigma}{2}\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) = \frac{\epsilon}{2},$$

and finish as before.

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INTRODUCTORY EXAMINATION—WINTER SESSION  
MORNING EXAM—LINEAR ALGEBRA

THURSDAY, JANUARY 19, 2023

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1. Let  $P_1, P_2 \in \mathbb{R}^{n \times n}$  be symmetric projection matrices. Show that if  $P_1 - P_2$  is a projection matrix, then  $\text{range}(P_2) \subseteq \text{range}(P_1)$ .

*Solution:*  $P_1 - P_2 = (P_1 - P_2)^2 = P_1 - P_1P_2 - P_2P_1 + P_2$ , so  $P_1P_2 + P_2P_1 = 2P_2$ . Then  $(I - P_1)P_2 + P_2(I - P_1) = 2P_2 - P_1P_2 - P_2P_1 = 0$ . Suppose  $x \in \text{range}(P_2) \setminus \text{range}(P_1)$ . Then  $(I - P_1)x \neq 0$ , and  $P_2(I - P_1)x = -(I - P_1)P_2x = -(I - P_1)x$ , which is impossible since the only possible eigenvalues of a projection matrix are 0 and 1.

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2. Find the rank of

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

*Solution:* The submatrix formed by rows 2-5 and columns 1-4 is invertible, since it is upper triangular with all diagonal entries equal to 1, so the rank of this matrix is at least 4. Now observe that the second column minus the first, and the fourth column minus the fifth, are equal, so the nullspace has dimension at least one. Put differently,  $[-1, 1, 0, -1, 1]^T \in \text{null}(A)$ . Then the rank must be exactly 4, by the rank-nullity theorem.

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3. Consider the matrix  $A$  below:

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$

If  $A$  is diagonalizable, find a diagonalizing matrix  $S$  such that  $S^{-1}AS$  is diagonal. Otherwise, show that  $A$  is not diagonalizable.

*Solution:*  $A$  is upper triangular, so its eigenvalues are equal to its diagonal entries. Since these are distinct,  $A$  is diagonalizable. We can clearly see that  $Ae_1 = 2e_1$ , so we want to find  $x, y$  such that

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since  $y$  can be anything, we pick  $y = 1$ , giving  $2x + 3 = x$ , or  $x = -3$ . Then

$$AS = A \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = SD,$$

giving the desired decomposition. Note that  $S$  is invertible since it is upper triangular with positive diagonal entries.

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4. Is there an invertible matrix  $A \in \mathbb{R}^{n \times n}$  for any value of  $n \geq 2$  for which every  $(n-1) \times (n-1)$  minor (not necessarily principal) is equal to 1? Prove that no such matrix can exist, or find a matrix with this property.

*Solution:* There is no such matrix. The cofactor formula for the inverse of a matrix  $A$  is  $(A^{-1})_{i,j} = (-1)^{i+j} \det(A[\{j\}^c, \{i\}^c]) = (-1)^{i+j}$ , since all  $(n-1) \times (n-1)$  minors equal 1. This would mean that  $A^{-1} = vv^T$  where  $v = [(-1)^i]_{i=1}^n$ , which is a rank one matrix, and cannot be invertible.

Alternate solution: Let  $e_1, e_2$  denote the first two standard unit vectors. Cramer's rule says that the solution to the equation  $Ax = e_1$  is given by  $x_i = (-1)^{i-1} / \det(A)$ , and  $Ay = e_2$  is given by  $y_i = (-1)^i / \det(A) = -x_i$ . Then  $e_1 = Ax = -A(-x) = -Ay = -e_2$ , which is impossible.

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5. Consider the real vector space  $V$  of polynomials with real coefficients having degree at most  $d$ , where  $d \geq 2$  is an integer. Show that the subspaces  $S_1, S_2$  satisfy  $\dim(S_1 \cap S_2) \geq (d-1)/2$ , where these are defined as:
- $S_1 = \{p \in V : p(1) = 0\}$ .
  - $S_2 = \{p \in V : p \text{ is even, i.e., } p(x) = p(-x) \text{ for all } x\}$ .

*Solution:* We observe that a polynomial  $p \in V$  may be written as

$$p(x) = \sum_{k=0}^d c_k x^k,$$

where  $c_k \in \mathbb{R}$  for all  $k$ . Then  $S_1$  may equivalently be defined as the set of polynomials such that  $\langle c, e \rangle = 0$ , where  $c = [c_k]_{k=0}^d$  and  $e = [1]_{k=0}^d$ . This is a subspace of dimension  $d$ . On the other hand, a polynomial is even if and only if  $c_k = 0$  for all odd  $k$ , so

$S_2$  is a subspace with dimension  $\lceil (d+1)/2 \rceil$ . By the subspace intersection lemma,  $\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - (d+1) = d + \lceil (d+1)/2 \rceil - (d+1) \geq (d-1)/2$ .

Alternate solution: Consider  $p_k(x) = x^{2k} - 1$ . Then  $\{p_1, p_2, \dots, p_{\lfloor d/2 \rfloor}\}$  are linearly independent, since they have different degrees, and belong to  $S_1 \cap S_2$ . When  $d$  is even, this gives a collection of  $d/2 \geq (d-1)/2$  linearly independent vectors in the subspace, and when  $d$  is odd, this gives  $(d-1)/2$  linearly independent vectors in the subspace.

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