Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SEMESTER
REAL ANALYSIS

Tuesday, August 20, 2024

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. What is the value of
$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}?$$

Solution: Observe that the function
$$f(x) \coloneqq \frac{1}{x(\ln x)(\ln \ln x)}, \quad x \in (e, +\infty),$$
is positive and strictly decreasing. In particular, $f(x) \leq f(\lfloor x \rfloor)$ on $(e, +\infty)$. Therefore for any $N \in \mathbb{N}$ such that $N > e$,
$$\sum_{n=3}^{N} \frac{1}{n \ln n \ln \ln n} = \int_3^{N+1} f(\lfloor x \rfloor)\, dx \geq \int_3^{N+1} f(x) dx.$$
Then we have
$$\int_3^{N+1} f(x) dx = \int_{\ln 3}^{\ln(N+1)} \frac{1}{y \ln y} dy = \int_{\ln \ln 3}^{\ln \ln (N+1)} \frac{dz}{z} = \ln \ln \ln (N+1) - \ln \ln \ln 3.$$
Therefore,
$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln \ln n} = \lim_{N \to \infty} \sum_{n=3}^{N} \frac{1}{n \ln n \ln \ln n} \geq \lim_{N \to \infty} \int_3^{N+1} f(x) dx = +\infty.$$

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2. Let $f : [0, 1] \to \mathbb{R}$ be of class $\mathcal{C}^1$ with $f(0) = 0$. Prove that
$$\|f\|_{\infty}^2 \leq \int_0^1 |f'(x)|^2 dx$$
where $\|f\|_{\infty} \coloneqq \sup \{|f(x)| : 0 \leq x \leq 1\}$.

Solution: Since $f \in \mathcal{C}^1([0, 1]; \mathbb{R})$ and $f(0) = 0$, the fundamental theorem of calculus implies that
$$f(x) = \int_0^x f'(y) dy, \quad \forall x \in [0, 1].$$
Then for all $x \in [0, 1]$ we have
$$|f(x)| = \left| \int_0^x f'(y) dy \right| \leq \int_0^x |f'(y)| dy \leq \int_0^1 |f'(y)| dy \implies \|f\|_{\infty} \leq \int_0^1 |f'(x)| dx.$$
The conclusion follows from the Cauchy–Schwarz inequality.

3. Consider the following subset of $\mathbb{R}^2$:

$$E \coloneqq \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0,1] \right\}.$$

Write down, without proof, the closure of E, denoted by $\overline{E}$. Then, prove or disprove: $\overline{E}$ is path-connected, i.e., for any pair of distinct points $p, q \in \overline{E}$, there exists a continuous function $\gamma : [a,b] \to \overline{E}$, with $-\infty < a < b < +\infty$, such that $\gamma(a) = p$ and $\gamma(b) = q$.

Solution: The closure of E is given by the disjoint union

$$\overline{E} = E \sqcup \{(0,y) : y \in [-1,1]\}.$$

Claim. $\overline{E}$ is not pathwise-connected.

Proof. First, observe that $E = \{(x, f(x)) : x \in (0,1)\}$ where $(0,1] \ni x \overset{f}{\mapsto} \sin \frac{1}{x}$. Assume, to the contrary, that $\overline{E}$ is pathwise-connected. Then there exists a continuous path $\gamma : [a,b] \to \overline{E}$, with $-\infty < a < b < +\infty$, such that $\gamma(a) = (0,0)$ and $\gamma(b) = (1, \sin 1)$. Notice that

$$[a,b] \ni t \overset{\gamma}{\mapsto} (x(t), y(t)) \in \overline{E}$$

being continuous implies that both x and y are continuous. Moreover, by the composition of $\overline{E}$, for all $t \in [a,b]$ such that $x(t) \in (0,1]$ we have $y(t) = f(x(t))$.

Define

$$\overline{a} \coloneqq \sup\{t \in [a,b] : x(t) = 0\}.$$

By continuity of x and the definition of supremum, we have that $x(\overline{a}) = 0$ and therefore $\overline{a} < b$. Write $y^o = y(\overline{a}) \in [-1,1]$ and take $y' \in [-1,1] \setminus \{y^o\}$. Denote

$$\bar{z}' = \arcsin y', \quad z' = \bar{z}' + 2\pi \mathbb{1}\{\bar{z}' < 1\}.$$

Let $x_n = \frac{1}{z' + 2(n-1)\pi}$ for all $n \in \mathbb{N}^+$. Then $x_n > x_{n+1}$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} x_n \downarrow 0$. If $z' = 1$ then take $c = b$; otherwise, by the intermediate value theorem, there exists $c \in (\overline{a}, b)$ such that $x(c) = \frac{1}{z'}$. Take $c_1 = c$ and we have $x(c_1) = x_1$; applying the intermediate value theorem repeatedly for all $n \in \mathbb{N}^+$, we can show that there exists $c_{n+1} \in (\overline{a}, c_n)$ such that $x(c_{n+1}) = x_{n+1}$. Thus,

$$\gamma(c_n) = (x(c_n), y(c_n)) = (x_n, f(x_n)) = (x_n, y'), \quad \forall n \in \mathbb{N}^+.$$

Observe that $\{c_n\}_{n\geq 1}$ is a strictly decreasing sequence in $[\overline{a}, b]$. By continuity of x and by definition of $\overline{a}$, we have $c_n \downarrow \overline{a}$ as $n \to \infty$. By continuity of y, we have $y(c_n) \xrightarrow{n\to\infty} y^o$. However, $y(c_n) \equiv y' \neq y^o$. The claim is concluded by contradiction.

Alternative proof:

The set $\overline{E}$ is the disjoint union of E and $\{0\} \times [-1, 1]$. Observe that for any $t > 0$ we have $E \cap (\{t\} \times \mathbb{R})$ consists of a single point $(t, \sin(1/t))$.

Suppose $\overline{E}$ is path-connected. so that there exists a continous function $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \overline{E}$ with $\gamma(0) = (2/\pi, 0)$ and $\gamma(1) \in F := \{0\} \times [-1, 1]$. If such a γ exists, then $\gamma^{-1}(F)$ is a closed non-empty subset of $[0, 1]$ and is hence compact. So there must exist

$$t^* = \min\{t \in [0, 1] : \gamma(t) \in F\},$$

and since $\gamma(0) \notin F$ we must have $t^* > 0$. Thus

(i) $\gamma_1(t) > 0$ for all $t \in [0, t^*)$ (since $\gamma(t) \notin F$ for $t < t^*$),

(ii) $\gamma_1(t^*) = 0$ (since $\gamma(t^*) \in F$), and

(iii) $\lim_{t\to t*} \gamma_1(t^*) = 0$.

For any $\epsilon > 0$, pick $n > \frac{1}{2\pi\epsilon}$ with $0 = \gamma_1(t^*) < \frac{1}{2\pi n+(\pi/2)} < \gamma_1(t^* - \epsilon)$. By the intermediate value there exists $t \in [t^* - \epsilon, t^*)$ with $\gamma_1(t) = \frac{1}{2\pi n+(\pi/2)}$. Since $\gamma(t) \in \overline{E}$ and $\gamma_1(t) > 0$ we have $\gamma_2(t) = \sin(1/\gamma_1(t_n)) = \sin(2\pi n + (\pi/2)) = 1$. Let $d(p, q)$ denote the Euclidean distance between $p, q \in \mathbb{R}^2$. So

$$d(\gamma(t), (0, 1)) = d\left(\left(\frac{1}{2\pi n + (\pi/2)}, 1\right), (0, 1)\right) = \frac{1}{2\pi n + (\pi/2)} < \frac{1}{2\pi n} < \epsilon.$$

By continuity of γ we must have $\lim_{t\to t^*} \gamma(t) = (0, 1)$.

Using the same argument as above, we can find $s \in [t^* - \epsilon, t^*)$ with $\gamma_1(s) = \frac{1}{2\pi n+(3\pi/2)}$ and hence $\gamma_2(s) = -1$ leading the the conclusion that $\lim_{s\to t^*} \gamma(s) = (0, -1)$, so we have arrived at a contradiction.

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4. Consider a sequence $(f_n)_{n\geq 1}$ of functions with $f_n \in \mathcal{C}^0([a, b]; \mathbb{R})$ for all $n \in \mathbb{N}^+$. Suppose that for any $x \in [a, b]$ we have $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = 0$. Prove or disprove: As $n \to \infty$, the functions f_n converge uniformly to 0 on $[a, b]$.

Solution: Claim: As $n \to \infty$, the functions f_n converge uniformly to 0 on $[a, b]$.

Proof. For any x we have $f_n(x) \to 0$ and $f_1(x) \geq f_2(x) \geq \cdots$, so $f_n(x) \geq 0$ for every n.

Fix $\epsilon > 0$. For any x there exists n_x such that $f_{n_x}(x) < \epsilon/2$, and by continuity of f_{n_x} there exists $\delta_x > 0$ such that $f_{n_x}(y) < \epsilon$ for all $y \in B(x, \delta_x)$. The balls $B(x, \delta_x)$ cover the compact interval $[a, b]$ so there exist $x_1, \ldots, x_k$ such that $[a, b] \subseteq \bigcup_{i=1}^k B(x_i, \delta_{x_i})$. Let $N := \max_{1 \le i \le k} n_{x_i}$. Then for any $x \in [a, b]$ there exists i with $x \in B(x_i, \delta_{x_i})$ and therefore, for $n \ge N$,

$$0 \le f_n(x) \le f_N(x) \le f_{n_{x_i}}(x) < \epsilon. \quad \square$$

Alternative Proof. For all $n \in \mathbb{N}^+$, the function f_n is continuous on $[a, b]$ and therefore $M_n = \sup_{x \in [a,b]} f_n(x)$ is attainable; choose any $x_n \in \arg\max_{x \in [a,b]} f_n(x)$ and we have $M_n = f_n(x_n)$. It suffices to establish $\lim_{n \to \infty} M_n = 0$.

Fix any $x \in [a, b]$. We have $0 \le f_n(x) \le M_n$ for any $n \in \mathbb{N}^+$. Also, observe that $M_{n+1} = f_{n+1}(x_{n+1}) \le f_n(x_{n+1}) \le M_n$ for any n and therefore $\{M_n\}_{n \ge 1}$ is a non-increasing sequence over $[0, M_1]$. Then $\lim_n M_n =: M \in [0, M_1]$.

Suppose that $M > 0$. Then there exists $\epsilon > 0$ such that $M > 3\epsilon$. Since $(x_n)_{n \ge 1} \subset [a, b]$, there exists a convergent subsequence $(x_{k_n})_{n \ge 1}$ such that $\lim_{n \to \infty} x_{k_n} = x_0 \in [a, b]$ by compactness of $[a, b]$ on $\mathbb{R}$.

On one hand, $f_n(x_0) \downarrow 0$ implies $f_{k_n}(x_0) \downarrow 0$. Then, there exists $N_1 \in \mathbb{N}^+$ such that for all $n \ge N_1$ we have $f_{k_n}(x_0) < \epsilon$. Denote $K := k_{N_1}$. By continuity of f_K, there exists $\delta > 0$ such that for all $x \in (-\delta + x_0, \delta + x_0)$ we have $|f_K(x) - f_K(x_0)| < \epsilon$, which implies $f_K(x) < 2\epsilon$. Since $x_{k_n} \xrightarrow{n \to \infty} x_0$, there exists $N_2 \in \mathbb{N}^+$ and $N_2 > N_1$ such that $|x_{k_n} - x| < \delta$ for all $n \ge N_2$, and therefore $f_K(x_{k_n}) < 2\epsilon$. Since $k_n > k_{N_1} = K$ for all $n \ge N_2 > N_1$, we have

$$M_{k_n} = f_{k_n}(x_{k_n}) \le f_K(x_{k_n}) < 2\epsilon, \quad \forall n \ge N_2.$$

On the other hand, $M_{k_n} \xrightarrow{n \to \infty} M > 3\epsilon$ therefore, there exists $N_3 \in \mathbb{N}^+$ and $N_3 > N_2$ such that for all $n \ge N_3$ we have $|M_{k_n} - M| < \epsilon$, which implies $M_{k_n} > M - \epsilon > 2\epsilon$. A contradiction occurs, and hence $\lim_{n \to \infty} M_n = 0$, which implies $f_n \rightrightarrows 0$ as $n \to \infty$. $\square$

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5. Let $d \in \mathbb{N}^+$, and consider a function $f : \mathbb{R}^d \to \mathbb{R}$ that satisfies
   - for every compact $K \subset \mathbb{R}^d$, $f(K)$ is a compact subset of $\mathbb{R}$; and
   - for every nested decreasing sequence of compact subsets $\{K_n\}_{n \ge 1}$ of $\mathbb{R}^d$,

$$f\left(\bigcap_{n \ge 1} K_n\right) = \bigcap_{n \ge 1} f(K_n).$$

Prove that f is continuous.

Solution: For any $x \in \mathbb{R}^d$, let $K_n = \overline{B}_{1/n}(x) := \{y \in \mathbb{R}^d : \|y - x\| \le \frac{1}{n}\}$ for all $n \in \mathbb{N}^+$, where $\|\cdot\|$ denotes the Euclidean norm. Observe that $x \in K_n$ and $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}^+$. For any $y \in \mathbb{R}^d \setminus \{x\}$, there exists $N \in \mathbb{N}^+$ such that $\|y - x\| > \frac{1}{n}$ for all $n \ge N$, thus $\{x\} = \bigcap_{n\ge1} K_n$.

The Heine–Borel theorem implies that the sets K_n are compact; thus so are the sets $f(K_n)$ by the first assumption about f. Using the second assumption, we find

$$\{f(x)\} = f\left(\bigcap_{n\ge1} K_n\right) = \bigcap_{n\ge1} f(K_n).$$

By the compactness of the sets $f(K_n)$, there exist $-\infty < m_n \le M_n < \infty$ such that $m_n, M_n \in f(K_n)$ and $f(K_n) \subset [m_n, M_n]$ for all n. Moreover, $K_{n+1} \subset K_n$ implies $f(K_{n+1}) \subset f(K_n)$. Therefore, (m_n) is non-decreasing and (M_n) is non-increasing; and for any $L \in \mathbb{N}^+$, the sequences $(m_n)_{n\ge L}$ and $(M_n)_{n\ge L}$ take values in the compact set $f(K_L)$. Notice that

$$m_1 \le m_2 \le \cdots \le f(x) \le \cdots \le M_2 \le M_1.$$

Then both $\lim_{n\to\infty} m_n$ and $\lim_{n\to\infty} M_n$ exist and belong to $\bigcap_{L\ge1} f(K_L)$. Thus, $f(x) = \lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n$. That is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}^+$, and thus $\delta = \frac{1}{N+1} > 0$, such that for all $y \in \mathbb{R}^d$ with $\|y - x\| < \delta$,

$$|f(y) - f(x)| \le M_n - m_n < \epsilon,$$

where $n = \lceil \frac{1}{\|y-x\|} \rceil > N$. Hence f is continuous at x. Finally, since x is arbitrarily fixed, f is continuous on $\mathbb{R}^d$.

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6. Show that the equation $xe^y + ye^x = 0$ defines implicitly a function $y = g(x)$ near the point $(0, 0)$ where g is of class $\mathcal{C}^\infty$. Compute $g^{(3)}(0)$ by using the chain rule.

Solution: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = xe^y + ye^x$. Observe that f is of class $\mathcal{C}^\infty$ and $f(0, 0) = 0$. The partial derivative

$$\partial_y f(x, y) = xe^y + e^x$$

does not vanish at $(0, 0)$, and hence the implicit function theorem can be applied to conclude that there exists a function g defined on a neighborhood of 0 such that $f(x, g(x)) = 0$; with f being $\mathcal{C}^\infty$, so is g.

Let $h(x) = f(x, g(x)) = 0$ and $k(x) = e^{g(x)}$. By the chain rule, for any $d \in \mathbb{N}^+$ we have

$$0 = h^{(d)}(x) = xk^{(d)}(x) + dk^{(d-1)}(x) + \sum_{l=0}^{d} \binom{d}{l} g^{(l)}(x)e^x;$$

at $x = 0$, we have $g(0) = 0$ and

$$0 = h^{(d)}(0) = dk^{(d-1)}(0) + \sum_{l=1}^{d} \binom{d}{l} g^{(l)}(0).$$

Observe that

$$k^{(0)}(x) = k(x) = e^{g(x)} \implies g^{(1)}(0) = -k^{(0)}(0) = -1;$$
$$k^{(1)}(x) = g^{(1)}(x)e^{g(x)} \implies g^{(2)}(0) = -2k^{(1)}(0) - 2g^{(1)}(0) = -4g^{(1)}(0) = 4;$$
$$k^{(2)}(x) = \{g^{(2)}(x) + [g^{(1)}(x)]^2\}e^{g(x)} \implies g^{(3)}(0) = -3k^{(2)}(0) - 3g^{(2)}(0) - 3g^{(1)}(0) = -24.$$

Department of Applied Mathematics and Statistics
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Introductory Examination–Fall Semester
Probability

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1. Jeff and Donna have three children. Two children are chosen uniformly at random on each day of the week to help with the dishes (with independent selections across days). What is the probability that at least one child gets chosen every day of the week?

   FYI: There are seven (7) days in a week, and you do not need to simplify your answer to a fraction.

   Solution: If we call the children A, B, and C, and let A_i (resp., B_i, C_i) be the event child A (resp., B, C) is selected on day i, $i = 1, 2, \ldots, 7$, then

$$A := \bigcap_{i=1}^{7} A_i, \quad B := \bigcap_{i=1}^{7} B_i, \quad C := \bigcap_{i=1}^{7} C_i$$

   represent the events that child A, B, C is selected all 7 days, respectively. Therefore, $A \cup B \cup C$ represents the event that at least one of A, B or C is selected all 7 days. By the inclusion–exclusion rule, and since $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C)$ and $\mathbb{P}(A \cap B) = \mathbb{P}(A \cap C) = \mathbb{P}(B \cap C)$ and $\mathbb{P}(A \cap B \cap C) = 0$, it follows that $\mathbb{P}(A \cup B \cup C) = 3\mathbb{P}(A) - 3\mathbb{P}(A \cap B) = 3(\frac{2}{3})^7 - 3(\frac{1}{3})^7 = \frac{127}{729} \doteq 0.174$.

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2. You are dealt two cards from a well-shuffled deck of 52 cards. Given that both cards are of the same color, what is the probability that they are of the same rank?

   FYI: The 52 cards are comprised of 13 ranks (2,3,4,5,6,7,8,9,10,J,Q,K,A) in each of 4 suits ($\clubsuit, \diamondsuit, \heartsuit, \spadesuit$), where $\clubsuit$ and $\spadesuit$ are colored black and $\diamondsuit$ and $\heartsuit$ are colored red.

   Solution: Out of the 25 remaining cards of the same color only 1 is the same rank; therefore, the desired conditional probability is $\frac{1}{25}$.

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3. Let X be a continuous random variable with probability density function (PDF) $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < +\infty$; such a random variable is said to have the *standard Cauchy distribution.*

   Find the PDF of the random variable $Y = \dfrac{1}{X}$.

Solution: Solution 1. The cumulative distribution function (CDF) F_Y of Y is given by

$$F_Y(y) = \mathbb{P}(Y \le y) = \begin{cases} \mathbb{P}(X < 0) + \mathbb{P}\big(X \ge \frac{1}{y}\big) = \frac{3}{2} - F_X\big(\frac{1}{y}\big), & y > 0; \\ \frac{1}{2}, & y = 0; \\ \mathbb{P}(\frac{1}{y} \le X < 0) = \frac{1}{2} - F_X(\frac{1}{y}), & y < 0, \end{cases}$$

with F_X being the CDF of X. On $\mathbb{R} \setminus \{0\}$, F_Y is clearly differentiable with $F_Y'(y) = \frac{1}{y^2} f(\frac{1}{y}) = f(y)$ by the chain rule. Also, it is easy to check, since f is even, that

$$F_Y'(0^+) = F_Y'(0^-) = \lim_{M \to +\infty} M \int_M^{+\infty} f(x)dx = f(0),$$

where the last equality follows from l'Hopital's rule. Therefore, F_Y is differentiable on $\mathbb{R}$ and the PDF of Y is $f_Y = f$; we see Y is, again, standard Cauchy.

Solution 2. Using the method of Jacobians: When $xy \neq 0$, $y = \frac{1}{x} \implies x = \frac{1}{y}$ is the inverse transformation, and $J = \frac{dx}{dy} = -\frac{1}{y^2}$, and

$$f_Y(y) = f(x) \cdot |J| = f\left(\frac{1}{y}\right) \cdot \frac{1}{y^2} = \frac{1}{\pi(1+y^2)} = f(y).$$

By definition,

$$f_Y(0) = \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(-\epsilon < Y < \epsilon)}{2\epsilon} = \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(|X| > \frac{1}{\epsilon})}{2\epsilon} = \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(X > \frac{1}{\epsilon})}{\epsilon} = f(0)$$

by the symmetry of f and l'Hopital's rule. Hence $f_Y = f$ on $\mathbb{R}$.

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4. Let X_1 and X_2 be independent standard normal random variables, and let U be a random variable that is uniformly distributed on the interval $[0, 1]$ and independent of both X_1 and X_2. We define $Z = UX_1 + (1-U)X_2$. Compute the mean and variance of Z.

Solution: We compute the conditional distribution of Z given $U = u$. In this case, $Z = uX_1 + (1-u)X_2$ is just a linear combination of independent Normal random variables, which is Normal. Observe that

$$\mathbb{E}(Z \,|\, U = u) = u\mathbb{E}(X_1 \,|\, U = u) + (1-u)\mathbb{E}(X_2 \,|\, U = u) = 0$$

and

$$\operatorname{Var}(Z \,|\, U = u) = u^2 \operatorname{Var}(X_1 \,|\, U = u) + (1-u)^2 \operatorname{Var}(X_2 \,|\, U = u) = u^2 + (1-u)^2.$$

Therefore, we have shown that $Z \,|\, U = u \sim N(0, u^2 + (1-u)^2)$. Finally, to finish the problem we invoke the laws of total expectation and total variance:

$$\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}[Z \,|\, U]) = \mathbb{E}(0) = 0,$$

and

$$\begin{aligned} \mathrm{Var}(Z) &= \mathrm{Var}(\mathbb{E}(Z \,|\, U)) + \mathbb{E}(\mathrm{Var}(Z \,|\, U)) \\ &= \mathrm{Var}(0) + \mathbb{E}[U^2 + (1-U)^2] = 0 + \mathbb{E}(U^2) + \mathbb{E}[(1-U)^2] = \tfrac{2}{3}. \end{aligned}$$

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5. Suppose $X_1, \ldots, X_n$ are independent and identically distributed random variables taking positive values. Find
$$\mathbb{E}\left[\frac{X_1 + \cdots + X_k}{X_1 + \cdots + X_n}\right]$$
for $k = 1, \ldots, n$.

Solution: Define $R_i := \frac{X_i}{X_1+\cdots+X_n}$ for $i = 1, \ldots, n$. Since the X_i are positive, R_i takes its value in $(0,1)$ and consequently $0 < \mathbb{E}[R_i] < 1 < +\infty$. In addition, the R_i are identically distributed, so $\mathbb{E}[R_i]$ does not depend on i. Since $\sum_{i=1}^n R_i \equiv 1$ we have $1 = \mathbb{E}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \mathbb{E}[R_i]$ so $\mathbb{E}[R_i] = 1/n$. We conclude that

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_k}{X_1 + \cdots + X_n}\right] = \sum_{i=1}^k \mathbb{E}\left[\frac{X_i}{X_1 + \cdots + X_n}\right] = \sum_{i=1}^k \mathbb{E}[R_i] = \frac{k}{n}.$$

Alternative solution:

For any i_j, $j = 1, \ldots, k$, satisfying $1 \le i_1 < \cdots < i_k \le n$, define

$$R_{i_i,\ldots,i_k} := \frac{\sum_{j=1}^k X_{i_j}}{\sum_{i=1}^n X_i}.$$

Since the X_i's are i.i.d., the $R_{i_i,\ldots,i_k}$'s are identically distributed. Then, by the linearity of expectation,

$$\begin{aligned} \mathbb{E}\left[\frac{X_1 + \cdots + X_k}{X_1 + \cdots + X_n}\right] &= \frac{\sum_{i_1,\ldots,i_k} \mathbb{E}[R_{i_i,\ldots,i_k}]}{\binom{n}{k}} = \mathbb{E}\left[\frac{\sum_{i=1}^n \binom{n-1}{k-1} X_i}{\binom{n}{k} \sum_{i=1}^n X_i}\right] \\ &= \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}. \end{aligned}$$

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6. Suppose X is a continuous random variable having probability density function $f(x) = e^{-x}$ for $x > 0$ (with $f(x) = 0$ for $x \le 0$). Compute

$$\mathbb{P}(\lfloor X \rfloor = n \text{ and } X - \lfloor X \rfloor \le x)$$

for $n \in \{0, 1, \ldots\}$ and $x \in [0, 1]$. Then answer with justification: Is it true that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are statistically independent?

FYI: $\lfloor X \rfloor$ is the greatest integer less than or equal to X.

Solution: For integer $n \ge 0$ and $x \in [0, 1]$ we have

$$\mathbb{P}(\lfloor X \rfloor = n \text{ and } X - \lfloor X \rfloor \le x) = \int_n^{n+x} e^{-u}\, du = e^{-n} - e^{-n-x} = (1 - e^{-x})e^{-n}.$$

Call this result (*). Setting $x = 1$ in (*) we find

$$\mathbb{P}(\lfloor X \rfloor = n) = (1 - e^{-1})e^{-n},$$

and summing (*) over n we find

$$\begin{aligned} \mathbb{P}(X - \lfloor X \rfloor \le x) &= \sum_{n=0}^{\infty} \mathbb{P}(\lfloor X \rfloor = n \text{ and } X - \lfloor X \rfloor \le x) \\ &= \sum_{n=0}^{\infty} (1 - e^{-x})e^{-n} = (1 - e^{-x}) \sum_{n=0}^{\infty} e^{-n} = \frac{1 - e^{-x}}{1 - e^{-1}}. \end{aligned}$$

Therefore, we've shown

$$\mathbb{P}(\lfloor X \rfloor = n \text{ and } X - \lfloor X \rfloor \le x) = \mathbb{P}(\lfloor X \rfloor = n)\, \mathbb{P}(X - \lfloor X \rfloor \le x);$$

so, yes, $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are statistically independent.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Linear Algebra

Thursday, August 22, 2024

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Suppose A is the $n \times n$ matrix all of whose columns are zero except for the i-th column, which is a column of 1's. If B is similar to A show that B is idempotent.

Solution: It is easy to check that $A^2 = A$. If B is similar to A we have $PBP^{-1} = A$ for some invertible matrix P and it follows that

$$PB^2P^{-1} = PBP^{-1}PBP^{-1} = A^2 = A = PBP^{-1}.$$

Multiplying on the left by P^{-1} and on the right by P yields $B^2 = B$.

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2. Suppose A is the $n \times (n+1)$ matrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

so that the i-th row of A has a 1 in positions i and $i+1$, for $i = 1, \ldots, n$, and the remaining entries of A are 0. Describe all possible solutions to the system of equations $Ax = y$, where y is the n-vector whose i-th entry is $2i - 1$ for $i = 1, \ldots, n$.

Solution: First, observe that $2i - 1 = (i-1) + i$ for all $i \in [n]$. Therefore,

$$x_p = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ n \end{pmatrix}$$

is a particular solution. Then by elementary column operations, A can be reduced to

$$\bar{A} = \begin{pmatrix} I_n & \vdots & b \end{pmatrix}$$

where

$$b = \begin{pmatrix} (-1)^{n-1} \\ (-1)^{n-2} \\ \vdots \\ 1 \end{pmatrix}.$$

Therefore the kernel of A is given by

$$\ker(A) = \left\{ c \begin{pmatrix} (-1)^0 \\ (-1)^1 \\ \vdots \\ (-1)^n \end{pmatrix} : c \in \mathbb{R} \right\}$$

and the set of solutions can be expressed as

$$\{x_p + x_0 : x_0 \in \ker A\}.$$

---

3. Suppose A is an $n \times n$ matrix with right inverse B. Show that B is also a left inverse A.

   Hint: For any x show that if $y = BAx$ then $y = x$.

   Solution: For any x, take $y = BAx$. We proceed to show that $y = x$ to give the conclusion that $BA = I$. Observe that $Ay = A(BAx) = (AB)Ax = IAx = Ax$ so $A(y - x) = 0$.. To complete the proof we just have to show that $\ker(A) = \{0\}$ i.e., that the columns of A are linearly independent.

   Let the columns of A be denoted by $a^{(1)}, \dots, a^{(n)}$. Since $AB = I$, we have

   $$b_{1i}a^{(1)} + \cdots + b_{ni}a^{(n)} = e^{(i)} \quad \text{for } i = 1, \dots, n.$$

   Since the vectors $e^{(i)}$, $i = 1, \dots, n$, span $\mathbb{R}^n$, the columns of A span $\mathbb{R}^n$ as well, and since there are n of them and $\dim(\mathbb{R}^n) = n$ they must be linearly independent.

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4. Define left-shift and right-shift transformations on $\mathbb{R}^n$ by

   $$L(x_1, x_2, \dots, x_{n-1}, x_n) = (x_2, x_3, \dots, x_{n-1}, x_n, 0)$$

   and

   $$R(x_1, x_2 \dots, x_{n-1}, x_n) = (0, x_1, x_2, x_3, \dots, x_{n-1}),$$

   and take V to be the vector space consisting of all linear combinations of compositions of these transformations.

   What is the dimension of V?

   Solution: Every composition of maps L and R sends the vector $(x_1, \dots, x_n)$ to a vector of the form

   $$(\underbrace{0, \dots, 0}_{\text{p}}, x_k, x_{k+1}, \dots, x_{k+m-1}, \underbrace{0, \dots, 0}_{\text{q}})$$

for some choice of nonnegative integers p, q, m, and k with $p+q+m=n$ and $1 \le k \le n-(m-1)$, and conversely, any such map can be expressed as a (right-to-left) composition of maps L and R, namely

$$R^p L^{n-m} R^{n-k-m+1}.$$

In particular, for every pair $1 \le j,k \le n$ there is a composition that maps $(x_1,\ldots,x_n)$ to

$$(\underbrace{0,\ldots,0}_{\text{k-1}}, x_j, \underbrace{0,\ldots,0}_{\text{n-k}}).$$

In terms of its action on the standard basis $\{e^{(1)},\ldots,e^{(n)}\}$ of $\mathbb{R}^n$, the space V contains every map $T_{j,k}$ defined by

$$T_{j,k}\left(\sum_{i=1}^n x_i e^{(i)}\right) = x_j e^{(k)},$$

i.e., by

$$T_{j,k}(e^{(i)}) = \begin{cases} e^{(k)} & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

for some choice of $1 \le j,k \le n$. Since any linear linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be expressed as a linear combination of such transformations, the dimension of V is n^2.

Alternative solution:

Define $L, R \in \mathbb{R}^{n\times n}$ as follows:

$$L = \begin{pmatrix} 0_{n-1,1} & I_{n-1} \\ 0 & 0_{1,n-1} \end{pmatrix}, \quad R = L^\top.$$

Then it is easy to check that for all $x \in \mathbb{R}^n$ we have

$$L(x) = Lx, \quad R(x) = Rx.$$

Therefore, $\dim(V) = \dim(\bar{V})$ where

$$\overline{V} = \operatorname{span}\left\{M_1 \cdots M_k : k \in \mathbb{N}^+ \text{ and } M_i \in \{L,R\} \text{ for all } i \in [k]\right\} \subseteq \mathbb{R}^{n\times n}.$$

For any $i,j \in [n]$, let $E_{i,j} = \Big(\mathbb{1}\{k=i \text{ and } l=j\}\Big)_{k,l}$; we know that $\mathbb{R}^{n\times n} = \operatorname{span}\{E_{i,j} : i,j \in [n]\}$. Notice that

$$\begin{aligned} L^{n-i}R^{n-1}L^{j-1} &= L^{n-i}R^{n-1}\begin{pmatrix} 0_{n-j+1,j-1} & I_{n-j+1} \\ 0_{j-1,j-1} & 0_{j-1,n-j+1} \end{pmatrix} \\ &= L^{n-i}\begin{pmatrix} 0_{n,j-1} & e^{(n)} & 0_{n,n-j} \end{pmatrix} \\ &= E_{i,j} = E_{j,i}^\top = \left(L^{n-i}R^{n-1}L^{j-1}\right)^\top = R^{j-1}L^{n-1}R^{n-i}, \end{aligned}$$

equivalent to the key observation in the previous solution. Therefore, $\overline{V} = \mathbb{R}^{n\times n}$ and $\dim(V) = n^2$.

5. Suppose $n \times n$ matrices A and B are simultaneously diagonalizable. Show that $AB = BA$.

Solution: Simultaneous diagonalizability gives the existence of an invertible matrix P such that $PAP^{-1} = D_A$ and $PBP^{-1} = D_B$ where D_A and D_B are diagonal matrices. Since diagonal matrices commute, we have

$$AB = P^{-1}D_A P P^{-1} D_B P = P^{-1} D_A D_B P = P^{-1} D_B D_A P = P^{-1} D_B P P^{-1} D_A P = BA.$$

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6. Suppose A is a $n \times n$ matrix for which $SAS^{-1} = \lambda A$ for some nonsingular $n \times n$ matrix S and $\lambda \neq 0$. Show that either $A^m = 0$ for some positive integer m or $\lambda^m = 1$ for some positive integer m.

   Hint: Consider the minimal polynomial of A.

Solution: For any nonnegative integer k we have

$$(SAS^{-1})^k = SA^k S^{-1}$$

and it follows that for any polynomial p we have

$$p(SAS^{-1}) = Sp(A)S^{-1}.$$

Taking $p(x) = x^m + c_{m-1}x^{m-1} + \cdots c_1 x + c_0$ to be the minimal polynomial of A, we have $p(A) = 0$ and $p(\lambda A) = p(SAS^{-1}) = Sp(A)S^{-1} = S0S^{-1} = 0$. Thus

$$A^m + c_{m-1}A^{m-1} + \cdots + c_1 A + c_0 I = 0 \tag{1}$$

and

$$\lambda^m A^m + c_{m-1}\lambda^{m-1}A^{m-1} + \cdots + c_1 \lambda A + c_0 I = 0. \tag{2}$$

Multiplying (1) by λ^m and subtracting (2) from it we obtain $q(A) = 0$, where

$$q(x) = c_{m-1}(\lambda^m - \lambda^{m-1})x^{m-1} + \cdots + c_1(\lambda^m - \lambda)x + c_0(\lambda^m - 1). \tag{3}$$

Since p is the minimal polynomial, q must be the zero polynomial, i.e., $c_j(\lambda^m - \lambda^j) = 0$ for $j = 0, \ldots, m-1$. If $c_j = 0$ for $j = 0, \ldots, m-1$, then $p(x) = x^m$ giving $A^m = 0$. Otherwise, $c_j \neq 0$ for some j, in which case $\lambda^m - \lambda^j = 0$ so $\lambda^{m-j} = 1$ since $\lambda \neq 0$.
