Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Semester
Real Analysis

Tuesday, August 20, 2024

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. What is the value of
$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}?$$

2. Let $f : [0,1] \to \mathbb{R}$ be of class $\mathcal{C}^1$ with $f(0) = 0$. Prove that
$$\|f\|_\infty^2 \leq \int_0^1 |f'(x)|^2 dx$$
where $\|f\|_\infty \coloneqq \sup\{|f(x)| : 0 \leq x \leq 1\}$.

3. Consider the following subset of $\mathbb{R}^2$:
$$E \coloneqq \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0,1] \right\}.$$
Write down, without proof, the closure of E, denoted by $\overline{E}$. Then, prove or disprove: $\overline{E}$ is path-connected, i.e., for any pair of distinct points $p, q \in \overline{E}$, there exists a continuous function $\gamma : [a,b] \to \overline{E}$, with $-\infty < a < b < +\infty$, such that $\gamma(a) = p$ and $\gamma(b) = q$.

4. Consider a sequence $(f_n)_{n \geq 1}$ of functions with $f_n \in \mathcal{C}^0([a,b]; \mathbb{R})$ for all $n \in \mathbb{N}^+$. Suppose that for any $x \in [a,b]$ we have $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = 0$. Prove or disprove: As $n \to \infty$, the functions f_n converge uniformly to 0 on $[a,b]$.

5. Let $d \in \mathbb{N}^+$, and consider a function $f : \mathbb{R}^d \to \mathbb{R}$ that satisfies
   - for every compact $K \subset \mathbb{R}^d$, $f(K)$ is a compact subset of $\mathbb{R}$; and
   - for every nested decreasing sequence of compact subsets $\{K_n\}_{n \geq 1}$ of $\mathbb{R}^d$,
   $$f\left(\bigcap_{n \geq 1} K_n \right) = \bigcap_{n \geq 1} f(K_n).$$

   Prove that f is continuous.

6. Show that the equation $xe^y + ye^x = 0$ defines implicitly a function $y = g(x)$ near the point $(0,0)$ where g is of class $\mathcal{C}^\infty$. Compute $g^{(3)}(0)$ by using the chain rule.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Semester
Probability

JF

Wednesday, August 21, 2024

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Jeff and Donna have three children. Two children are chosen uniformly at random on each day of the week to help with the dishes (with independent selections across days). What is the probability that at least one child gets chosen every day of the week?

   FYI: There are seven (7) days in a week, and you do not need to simplify your answer to a fraction.

2. You are dealt two cards from a well-shuffled deck of 52 cards. Given that both cards are of the same color, what is the probability that they are of the same rank?

   FYI: The 52 cards are comprised of 13 ranks (2,3,4,5,6,7,8,9,10,J,Q,K,A) in each of 4 suits ($\clubsuit, \diamondsuit, \heartsuit, \spadesuit$), where $\clubsuit$ and $\spadesuit$ are colored black and $\diamondsuit$ and $\heartsuit$ are colored red.

3. Let X be a continuous random variable with probability density function (PDF) $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < +\infty$; such a random variable is said to have the *standard Cauchy distribution.*

   Find the PDF of the random variable $Y = \dfrac{1}{X}$.

4. Let X_1 and X_2 be independent standard normal random variables, and let U be a random variable that is uniformly distributed on the interval $[0, 1]$ and independent of both X_1 and X_2. We define $Z = UX_1 + (1 - U)X_2$. Compute the mean and variance of Z.

5. Suppose $X_1, \ldots, X_n$ are independent and identically distributed random variables taking positive values. Find

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_k}{X_1 + \cdots + X_n}\right]$$

   for $k = 1, \ldots, n$.

6. Suppose X is a continuous random variable having probability density function $f(x) = e^{-x}$ for $x > 0$ (with $f(x) = 0$ for $x \leq 0$). Compute

$$\mathbb{P}(\lfloor X \rfloor = n \text{ and } X - \lfloor X \rfloor \leq x)$$

for $n \in \{0, 1, \ldots\}$ and $x \in [0, 1]$. Then answer with justification: Is it true that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are statistically independent?

FYI: $\lfloor X \rfloor$ is the greatest integer less than or equal to X.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Linear Algebra

Thursday, August 22, 2024

Instructions: Read carefully!

1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Suppose A is the $n \times n$ matrix all of whose columns are zero except for the i-th column, which is a column of 1's. If B is similar to A show that B is idempotent.

2. Suppose A is the $n \times (n+1)$ matrix of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

so that the i-th row of A has a 1 in positions i and $i+1$, for $i = 1, \ldots, n$, and the remaining entries of A are 0. Describe all possible solutions to the system of equations $Ax = y$, where y is the n-vector whose i-th entry is $2i - 1$ for $i = 1, \ldots, n$.

3. Suppose A is an $n \times n$ matrix with right inverse B. Show that B is also a left inverse of A.

   Hint: For any x show that if $y = BAx$ then $y = x$.

4. Define left-shift and right-shift transformations on $\mathbb{R}^n$ by

$$L(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_2, x_3, \ldots, x_{n-1}, x_n, 0)$$

   and

$$R(x_1, x_2 \ldots, x_{n-1}, x_n) = (0, x_1, x_2, x_3, \ldots, x_{n-1}),$$

   and take V to be the vector space consisting of all linear combinations of compositions of these transformations.

   What is the dimension of V?

5. Suppose $n \times n$ matrices A and B are simultaneously diagonalizable. Show that $AB = BA$.

6. Suppose A is a $n \times n$ matrix for which $SAS^{-1} = \lambda A$ for some nonsingular $n \times n$ matrix S and $\lambda \neq 0$. Show that either $A^m = 0$ for some positive integer m or $\lambda^m = 1$ for some positive integer m.

   Hint: Consider the minimal polynomial of A.