Department of Applied Mathematics and Statistics The Johns Hopkins University

INTRODUCTORY EXAMINATION–SPRING SEMESTER REAL ANALYSIS

Tuesday, January 16, 2024

Instructions: Read carefully!

- 1. This **closed-book** examination consists of 6 problems, each worth 5 points. Your best five scores will be used to determine the exam grade. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
- 2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear**, **logically justified steps**. The grading will reflect that broader purpose.
- 3. The problems have not been arranged systematically by difficulty.
- 4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
- 5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
- 6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
- 7. No calculators of any sort are needed or permitted.

1. Let $f:[0,1] \to \mathbb{R}$ be a Riemann integrable function satisfying

$$\int_0^1 |f(x)| \,\mathrm{d}x = 0.$$

- (a) Use the definition of continuity to show that if f is continuous then f = 0.
- (b) Give an example of a discontinuous function f that satisfies the assumptions of the problem but $f \neq 0$.
- 2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + y} & \text{if } x^2 \neq -y \\ 0 & \text{if } x^2 = -y. \end{cases}$$

Show that all directional derivatives at (0,0) exist but f is not differentiable at (0,0).

- 3. Let X be a topological space and consider a function $f : X \to X$. (You may not assume that X is a metric space for this problem.)
 - (a) Define what it means for f to be continuous.
 - (b) Let A be a subset of X. Define what it means for A to be compact.
 - (c) Show that f(A) is compact if f is continuous and A is compact.
- 4. Give an example of a closed and bounded set (in some metric space) that is not compact. Prove all your claims.
- 5. Show that the sequence $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, $\sqrt{2\sqrt{2}\sqrt{2}}$, ... converges and find its limit.
- 6. (a) By direct calculation, determine at precisely what points $(x_0, y_0) \in \mathbb{R}^2$ you can solve the equation $F(x, y) = y^2 + y + 3x + 1 = 0$ for y as a unique, continuously differentiable real-valued function f defined for x in a neighborhood of x_0 so that $y_0 = f(x_0)$.
 - (b) Check whether your answer to part (a) agrees with the answer you expect from the implicit function theorem. Compute dy/dx.

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INTRODUCTORY EXAMINATION–SPRING SEMESTER PROBABILITY

Wednesday, January 17, 2024

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- 2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear**, **logically justified steps**. The grading will reflect that broader purpose.
- 3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
- 4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
- 5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
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- 1. Suppose X_1 and X_2 are independent Poisson random variables with means λ_1 and λ_2 respectively. Determine the conditional distribution of X_1 given that $X_1 + X_2 = n$.
- 2. Let S_n be the number of successes in n tosses of a bent coin whose success probability is p. Prove a weak law of large numbers for $\frac{S_n}{n}$, i.e., show that $\frac{S_n}{n}$ converges to p in probability as $n \to \infty$.
- 3. We start with a stick of length L. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process with the stick that we keep. After breaking twice, what are the expected length and variance of the stick we are left with?
- 4. Let n > 3.
 - (a) Suppose a fair coin is flipped n times. Let p_n be the probability that there are at most 3 heads. Find p_n as a simple function of n.
 - (b) Suppose a fair coin is flipped until heads appears 3 times. Let q_n be the probability that it takes at least n flips (including the flip resulting in the 3rd head). Find q_n as a simple function of n.
 - (c) Which of the following holds: $p_n > q_n$, $p_n = q_n$, $p_n < q_n$? Justify your answer.
- 5. Let 0 and let X be a random variable such that

$$P(X = 1) = p, \quad P(X = -1) = 1 - p.$$

Let Y be a random variable whose conditional distribution given X is normal with mean X and variance 4, that is, $Y|X \sim N(X, 4)$.

- (a) Conditioned on Y = y, what is the distribution of X? Simplify your answer.
- (b) What is the variance of Y?
- 6. Let X, Y be random variables whose joint density function is

$$p(x,y) = 2e^{x-2y}, \quad x \ge 0, \ y \ge x.$$

Let U, V be independent random variables whose distributions are uniform over (0,1). Find an explicit continuous function $f: (0,1)^2 \to \mathbb{R}^2$ such that f(U,V) has the same distribution as (X,Y).

Department of Applied Mathematics and Statistics The Johns Hopkins University

INTRODUCTORY EXAMINATION–SPRING SEMESTER LINEAR ALGEBRA

THURSDAY, JANUARY 18, 2024

Instructions: Read carefully!

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- 5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
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- 7. No calculators of any sort are needed or permitted.

1. Consider the two real 3×3 matrices

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 5 \\ 0 & 5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -6 & 1 \\ -6 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Show that there is no basis for \mathbb{R}^3 which diagonalizes A and B simultaneously.

2. If A is a square matrix with all eigenvalues real, prove that the matrix $B = I + A + \frac{1}{2}A^2$ is invertible.

Hint: Consider the Jordan canonical form of A.

- 3. If W is the $n \times n$ matrix all of whose entries are ones and if I is the $n \times n$ identity matrix, find the inverse matrix $(I + W)^{-1}$.
- 4. If A is a linear map from a vector space S to a vector space T with $\dim(S) > \dim(T)$, then prove that the subspace of vectors $x \in S$ such that Ax = 0 has dimension at least $\dim(S) \dim(T)$.
- 5. Assume that u, v are two vectors in \mathbb{C}^n and consider the following possible assignments of Euclidean norms:

(i)
$$||u+v|| = 2$$
, $||u-v|| = 2$, $||u+iv|| = 3$, $||u-iv|| = 3$
(ii) $||u+v|| = 2$, $||u-v|| = 2$, $||u+iv|| = 2$, $||u-iv|| = 2$.

For each assignment, either prove that it is impossible or find an illustrative example.

6. If A is a Hermitian, positive-definite $n \times n$ matrix, B is an $n \times m$ matrix for $m \leq n$ with full rank, and O_m is the $m \times m$ zero matrix, then prove that the $(n+m) \times (n+m)$ matrix C defined by

$$C = \begin{pmatrix} A & B \\ B^* & O_m \end{pmatrix}$$

is Hermitian with all eigenvalues non-zero.