Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Real Analysis

Monday, August 21, 2023

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit. However, hints are optional and solutions that don't use the hints are welcome too.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. (a) Compute $\lim_{n\to\infty} \sqrt[n]{n}$.
   (b) For any $x \in \mathbb{R}$ show that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.
   Note: If you use the fact that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, then you must prove that this power series converges for all $x \in \mathbb{R}$.

Solution: (a) Taking the natural logarithm,

$$\lim_{n\to\infty} \ln(\sqrt[n]{n}) = \lim_{n\to\infty} \frac{\ln n}{n} = 0,$$

for example by l'Hôpital's rule. Thus, $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(b) For any $x \in \mathbb{R}$, take $N = \lceil |x| + 1 \rceil$. Then, for $n > N$,

$$\frac{|x|^n}{n!} = \frac{|x|^N}{N!} \prod_{k=N+1}^{n} \frac{|x|}{k} < \frac{|x|^N}{N!} \left(\frac{|x|}{|x|+1} \right)^{n-N}.$$

Since $\frac{|x|}{|x|+1} < 1$, then $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

Alternatively, one can prove a stronger result that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x < \infty$ for all $x \in \mathbb{R}$. Using Stirling's approximation $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$ we have

$$\frac{|x|}{\sqrt[n]{n!}} \sim \frac{|x|}{(2\pi n)^{\frac{1}{2n}} (n/e)}.$$

Since $\lim_{n\to\infty} (2\pi n)^{\frac{1}{2n}} = 1$ by part (a), it follows that $\lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n!}} = 0$ and thus the Taylor series has infinite radius of convergence by the root test.

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2. Prove that if $f : [a, b] \to \mathbb{R}$ is a monotonically decreasing function in the compact interval $[a, b]$, then f is Riemann integrable in $[a, b]$.

   Hint: One possible way to solve this problem is by using the following definition of Riemann integrable functions. For each uniform partition $x_i = a + i\Delta_n$, $i = 0, 1, ..., n$ with $\Delta_n = (b - a)/n$, consider the partial sums

$$U_n(f, a, b) := \sum_{i=0}^{n-1} \sup_{x\in[x_i, x_{i+1}]} f(x) \cdot \Delta_n, \quad L_n(f, a, b) := \sum_{i=0}^{n-1} \inf_{x\in[x_i, x_{i+1}]} f(x) \cdot \Delta_n$$

   and define the upper and lower Riemann integrals by

$$U(f, a, b) = \inf_n U_n(f, a, b), \quad L(f, a, b) = \sup_n L_n(f, a, b),$$

A function is said to be Riemann integrable if $L(f,a,b) = U(f,a,b)$. It may be helpful to show that $L(f,a,b) \le U(f,a,b)$ for all f and $\lim_{n\to\infty}[U_n(f,a,b) - L_n(f,a,b)] = 0$ for f monotone decreasing.

Solution: The result $L(f,a,b) \le U(f,a,b)$ is proved by noting that for any n, m

$$L_n(f,a,b) \le L_{nm}(f,a,b) \le U_{nm}(f,a,b) \le U_m(f,a,b) \tag{$*$}$$

and then taking the supremum over n and the infimum over m.

It is thus enough to show that $U(f,a,b) \le L(f,a,b)$. However, by monotonicity of f

$$U_n(f,a,b) = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta_n, \quad L_n(f,a,b) = \sum_{i=0}^{n-1} f(x_{i+1}) \cdot \Delta_n$$

and subtracting gives

$$U_n(f,a,b) - L_n(f,a,b) = (f(a) - f(b))\Delta_n = (f(a) - f(b))(b-a)/n.$$

Because of the monotonicity of $U_n(f,a,b)$ in n expressed by (*), we can always choose an increasing subsequence $n_k \uparrow \infty$ so that $\lim_{k\to\infty} U_{n_k}(f,a,b) = U(f,a,b)$ and thus

$$\begin{aligned} U(f,a,b) \ = \ \lim_{k\to\infty} U_{n_k}(f,a,b) \ &= \ \lim_{k\to\infty}[U_{n_k}(f,a,b) + (f(b) - f(a))(b-a)/n_k] \\ &= \ \lim_{k\to\infty} L_{n_k}(f,a,b) \ \le \ \sup_n L_n(f,a,b) \ = \ L(f,a,b). \end{aligned}$$

Alternative Solution: According to the Lebesgue-Vitali theorem, a bounded function on a compact interval $[a,b]$ is Riemann integrable if and only if it is continuous almost everywhere, i.e., the set of its points of discontinuity has Lebesgue measure zero. However, a monotone decreasing function f can have only a countable set $\mathcal{D}_f$ of points of discontinuity, since all of the intervals $(f(x+), f(x-))$ for $x \in \mathcal{D}_f$ are disjoint and contained in the interval $(f(b), f(a))$ so that

$$\sum_{x\in\mathcal{D}_f} (f(x-) - f(x+)) \le f(a) - f(b) < \infty.$$

Here the sum over $x \in \mathcal{D}_f$ is interpreted as the supremum of the sums over finite subsets of $\mathcal{D}_f$. Since a sum (in this sense) over an uncountable set of positive real numbers must be infinite, we conclude from the finite upper bound that the set $\mathcal{D}_f$ must be countable. Thus, a monotone decreasing f is Riemann integrable.

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3. Show that the equation $z^3 + 2z + \exp(z - x - y^2) = \cos(x - y + z)$ defines implicitly a function $z = f(x, y)$ near the point $(0, 0, 0)$ where f is infinitely differentiable. Compute the differential of f at $(0, 0)$.

Solution: Since

$$F(x, y, z) = z^3 + 2z + \exp(z - x - y^2) - \cos(x - y + z)$$

is infinitely differentiable (C^∞), the implicit function theorem implies that if

$$F_z(0, 0, 0) \neq 0,$$

then a C^∞ function f is defined near $(0, 0)$ so that $F(x, y, f(x, y)) = 0$ and

$$f_x(x, y) = -\frac{F_x(x, y, f(x, y))}{F_z(x, y, f(x, y))}, \ f_y(x, y) = -\frac{F_y(x, y, f(x, y))}{F_z(x, y, f(x, y))}.$$

Simple computations give

$$\begin{aligned} F_x(x, y, z) &= -\exp(z - x - y^2) + \sin(x - y + z) \\ F_y(x, y, z) &= -2y \exp(z - x - y^2) - \sin(x - y + z) \\ F_z(x, y, z) &= 3z^2 + 2 + \exp(z - x - y^2) + \sin(x - y + z) \end{aligned}$$

so that $F_z(0, 0, 0) = 3 \neq 0$ and

$$f_x(0, 0) = -(-1)/3 = 1/3, \ f_y(0, 0) = -(0)/3 = 0.$$

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4. Consider a sequence of nonnegative real numbers $a_n, n \geq 0$. Show that

$$\prod_{n=0}^{\infty}(1 + a_n) \text{ converges} \quad \text{if and only if} \quad \sum_{n=0}^{\infty} a_n \text{ converges.}$$

The following inequalities may be helpful, and can be used without proof: $x/2 \leq \log(1 + x) \leq x$ for $x \in [0, 2]$.

Solution: If $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$, so that $a_n \in [0, 2]$ for $n \geq N$ with some sufficiently large N. In that case, for any $M > N$ we have

$$\ln\left(\prod_{n=N}^{M}(1 + a_n)\right) = \sum_{n=N}^{M} \ln(1 + a_n) \leq \sum_{n=N}^{M} a_n \leq \sum_{n=N}^{\infty} a_n < \infty.$$

Since the sequence $p_M := \prod_{n=N}^{M}(1+a_n)$ is thus bounded and non-decreasing, it has a finite limit as $M \to \infty$.

Conversely, if $\prod_{n=0}^{\infty}(1+a_n)$ converges, then likewise $\lim_{n\to\infty} a_n = 0$, so that $a_n \in [0,2]$ for $n \geq N$ with some sufficiently large N. In that case, for any $M > N$ we have

$$\sum_{n=N}^{M} a_n \leq 2\sum_{n=N}^{M} \ln(1+a_n) = 2\ln\left(\prod_{n=N}^{M}(1+a_n)\right) \leq 2\ln\left(\prod_{n=N}^{\infty}(1+a_n)\right) < \infty.$$

Since the sequence $s_M := \sum_{n=N}^{M} a_n$ is thus bounded and non-decreasing, it has a finite limit as $M \to \infty$.

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5. Denote by $\mathcal{C}[0,1]$ the space of continuous functions $f : [0,1] \to \mathbb{R}$ and define the subset $\mathcal{F} = \{f \in \mathcal{C}[0,1] : \|f\| \leq 1\}$ with $\|f\| := \max_{x\in[0,1]} |f(x)|$. Show that $\mathcal{F}$ is closed and bounded in $\mathcal{C}[0,1]$ in the norm topology. Is $\mathcal{F}$ compact or, equivalently, sequentially compact?

   Hint: The Arzelà-Ascoli theorem is one way to answer the compactness question.

Solution: By its definition, $\mathcal{F}$ is clearly bounded by 1.

Furthermore, a sequence $f_n \in \mathcal{F}$ converges to f in $\mathcal{C}[0,1]$ if and only if it converges uniformly. In that case, for all $x \in [0,1]$, since $|f_n(x)| \leq 1$, one has $|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq 1$. Thus, $\|f\| = \max_{x\in[0,1]} |f(x)| \leq 1$ and $f \in \mathcal{F}$. Therefore, $\mathcal{F}$ is closed as well.

However, $\mathcal{F}$ is not compact, which is equivalent to sequential compactness for metric spaces. By the Arzelà-Ascoli theorem, $\mathcal{F}$ is sequentially compact if and only if $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous. Clearly, $f_n(x) = \sin(\pi n x)$ is a subsequence in $\mathcal{F}$ but it is not uniformly equicontinuous since $|f_n(0) - f_n(\frac{1}{2n})| = |0-1| = 1$ even though $\lim_{n\to\infty} \frac{1}{2n} = 0$.

Note here that we are using only the easier half of the Arzelà-Ascoli theorem, i.e., that uniform boundedness and uniform equicontinuity are necessary for sequential compactness and not the deeper sufficiency statement. In fact, to answer the compactness question in this problem we need to use only the fact that any sequence $f_n \in C[0,1]$ which converges uniformly to $f \in C[0,1]$ must be uniformly equicontinuous, and this result follows easily from the triangle inequality

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)|.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Probability

Tuesday, August 22, 2023

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7. **No calculators of any sort are needed or permitted.**

1. In a memory game, $2n$ cards containing the numbers $1, \ldots, n$, each appearing twice, are shuffled and placed face down on the table. At each turn, you may pick two cards, turn them face up, and if they match, remove them from the table. If they do not match, they are placed face down again. The game is finished when all cards are removed.

   Suppose that instead of trying to remember the cards, you pick two cards independently and uniformly at random each turn. (That is, each of the $\binom{2n}{2}$ pairs is equally likely to be chosen.) What is the expected number of turns before you finish the game?

   Solution: Let X_k denote the number of turns between the first time when there are $2k$ pairs on the table, and when the next pair is removed. When there are $2k$ pairs, the probability of picking one pair is $\frac{1}{2k-1}$ (no matter what the first card is, there is probability $\frac{1}{2k-1}$ that the other card will match). Thus X_k is a geometric random variable with success probability $\frac{1}{2k-1}$, and has expected value $2k-1$. The number of turns until the game finishes is $\sum_{k=1}^{n} X_k$. By linearity of expectation,

$$\mathbb{E}\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \mathbb{E}X_k = \sum_{k=1}^{n}(2k-1) = n^2.$$

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2. Suppose that X and Y are independent exponential random variables with rates λ and ν, respectively. The probability density function of an exponential random variable with rate λ is $\lambda e^{-\lambda x}$ for $x \geq 0$.

   Find the cumulative distribution function and probability density function of $\max\{X, Y\}$.

   Solution: Note that the cdf for X and Y are $1 - e^{-\lambda x}$ and $1 - e^{-\nu x}$, respectively. The cdf of $\max\{X, Y\}$ is

$$\begin{aligned} F(x) := \mathbb{P}(\max\{X, Y\} \leq x) &= \mathbb{P}(X \leq x, Y \leq x) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq x) \\ &= (1 - e^{-\lambda x})(1 - e^{-\nu x}), \quad x \geq 0, \end{aligned}$$

   and the pdf is

$$p(x) = F'(x) = \lambda e^{-\lambda x} + \nu e^{-\nu x} - (\lambda + \nu)e^{-(\lambda+\nu)x}, \quad x \geq 0,$$

   with $F(x) = 0$ and $p(x) = 0$ for $x < 0$.

3. A bag has one blue and one red ball. At each step, a random ball is drawn from the bag, with all balls equally likely to be chosen. The ball is then placed back in the bag with another ball of the same color. This is repeated independently 10 times. What is the probability that 4 red balls and 6 blue balls are drawn, in any order? Fully simplify your answer.

Solution: More generally, suppose that $r + b$ balls are drawn; we compute the probability that r red balls and b blue balls are drawn. First consider the probability that all the red balls are drawn first, and then all the blue balls are drawn. This probability equals

$$\frac{1}{2} \cdots \frac{r}{r+1} \cdot \frac{1}{r+2} \cdots \frac{b}{r+b+1} = \frac{r!b!}{(r+b+1)!}.$$

To see this, note that the kth time the ball of a color is drawn, there are k balls of that color in the bag; morever, the number of balls increases by 1 in each step. Next, note that for any particular ordering of r red balls and b blue balls, the probability is the same, as this only has the effect of permuting the numerators. Hence the total probability equals

$$\binom{r+b}{r} \frac{r!b!}{(r+b+1)!} = \frac{(r+b)!}{r!b!} \cdot \frac{r!b!}{(r+b+1)!} = \frac{1}{r+b+1}.$$

In our case, the answer is $\boxed{\frac{1}{11}}$.

4. For $\alpha, \beta > 0$, the Gamma(α, β) distribution is the probability distribution supported on $[0, \infty)$ with density function

$$p(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}.$$

(Here, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$, but the definition of Γ will not be necessary for this problem; you may leave answers in terms of Γ.) Suppose that U is drawn from the Gamma(α, β) distribution, we let $\sigma = U^{-1/2}$, and then $X_1, \ldots, X_n$ are drawn i.i.d. from the distribution $N(0, \sigma^2)$. Compute the density function of U conditional on $X_1 = x_1, \ldots, X_n = x_n$.

Solution: The conditional density function of $N(0, \sigma^2)$ given $U = u$ is proportional to $\frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} = u^{1/2} e^{-ux^2/2}$ for $x \in \mathbb{R}$. By Bayes's Rule,

$$\begin{aligned} p_{U|X_1,\dots,X_n}(u|x_1,\dots,x_n) &= \frac{p_U(u) p_{X_1,\dots,X_n|U}(x_1,\dots,x_n|u)}{p_{X_1,\dots,X_n}(x_1,\dots,x_n)} \\ &\propto u^{\alpha-1} e^{-\beta u} \cdot \prod_{i=1}^{n} \left(u^{1/2} e^{-ux_i^2/2} \right) \\ &\propto u^{(\alpha+\frac{n}{2})-1} e^{-u(\beta+\frac{1}{2}\sum_{i=1}^n x_i^2)} \end{aligned}$$

where we ignore constants that do not depend on u but may depend on $x_1, \dots, x_n$. We recognize this as the Gamma$(\alpha + \frac{n}{2}, \beta + \frac{1}{2}\sum_{i=1}^n x_i^2)$ distribution. Hence, including the normalizing constant for the Gamma distribution, the required density is

$$p_{U|X_1,\dots,X_n}(u|x_1,\dots,x_n) = \frac{(\beta + \frac{1}{2}\sum_{i=1}^n x_i^2)^{\alpha+\frac{n}{2}}}{\Gamma(\alpha + \frac{n}{2})} u^{(\alpha+\frac{n}{2})-1} e^{-u(\beta+\frac{1}{2}\sum_{i=1}^n x_i^2)}.$$

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5. At the Motor Vehicle Administration, there is an infinitely long line of people and a single clerk. The amount of time in minutes that each customer takes at the counter is an i.i.d. random variable with mean 10 minutes and standard deviation 2 minutes. Customers are served continuously, one after the other.

   Let $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ be the cumulative distribution function of a standard Gaussian random variable. Let S_n be the amount of time in minutes that it takes to serve the first n customers. Find a function f such that

   $$\lim_{n\to\infty} \mathbb{P}(S_n \le f(n)) = 0.05.$$

   Your answer may involve Φ^{-1}.

Solution: Let X_n be the amount of time taken by the nth customer. Then $S_n = X_1 + \cdots + X_n$. By the Central Limit Theorem,

$$\frac{S_n - 10n}{2\sqrt{n}} \xrightarrow{d} N(0,1).$$

Convergence in distribution means that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - 10n}{2\sqrt{n}} \le z \right) = \Phi(z).$$

Taking $z = \Phi^{-1}(0.05)$ gives that

$$\lim_{n\to\infty} \mathbb{P}\left(S_n \le 10n + 2\Phi^{-1}(0.05)\sqrt{n}\right) = \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - 10n}{2\sqrt{n}} \le \Phi^{-1}(0.05) \right) = 0.05.$$

Thus, we can take $f(n) = 10n + 2\Phi^{-1}(0.05)\sqrt{n}$.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–FALL SESSION
LINEAR ALGEBRA

WEDNESDAY, AUGUST 23, 2023

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1. Suppose that X is a real, 4×2 matrix and

$$XX^\top = \begin{pmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & ? & ? \\ 2 & -1 & ? & ? \end{pmatrix}.$$

Fill in the missing entries marked by "?".

Solution: Since $XX^\top$ must have rank 2, its last two columns must be linear combinations of the first two columns. The coefficients of the linear combinations are easily determined by the given entries as

$$\begin{pmatrix} 1 \\ 0 \\ ? \\ ? \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -1 \\ ? \\ ? \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 5 \end{pmatrix},$$

so that

$$XX^\top = \begin{pmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 5 \end{pmatrix}.$$

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2. Calculate the matrix exponential e^{tA} for real t and matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solution: It is easy to show by induction that

$$A^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}, \quad n \geq 0,$$

and thus

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = \begin{pmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{pmatrix}.$$

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3. Prove for any real $n \times m$ matrix A that $\mathrm{Ran}(A^\top A) = \mathrm{Ran}(A^\top)$, where $\mathrm{Ran}(B)$ denotes the range of a given matrix B.

Solution: The result follows from

$$\begin{aligned} \mathrm{Ran}(A^\top A) &= \{A^\top A x \mid x \in \mathbb{R}^m\} \\ &= \{A^\top y \mid y \in \mathrm{Ran}(A)\} \\ &= \{A^\top (y+z) \mid y \in \mathrm{Ran}(A),\ z \in \mathrm{Ker}(A^\top)\} \\ &= \{A^\top x \mid x \in \mathbb{R}^n\} \text{ since } \mathbb{R}^n = \mathrm{Ran}(A) \oplus \mathrm{Ker}(A^\top) \\ &= \mathrm{Ran}(A^\top). \end{aligned}$$

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4. If U is a unitary complex $n \times n$ matrix such that $I + U$ is invertible, then prove that

$$H = i(I+U)^{-1}(U-I)$$

is Hermitian. *Hint:* Show that $H^* = i(U-I)(U+I)^{-1}$.

Solution: Since

$$U^* - I = U^{-1} - I = (I-U)U^{-1}, \quad I + U^* = I + U^{-1} = (U+I)U^{-1},$$

then

$$H^* = -i(U^*-I)(I+U^*)^{-1} = -i(I-U)U^{-1} \cdot U(U+I)^{-1} = i(U-I)(U+I)^{-1}.$$

Since $U - I$ and $U + I$ commute, then so do $U - I$ and $(U+I)^{-1}$. Thus,

$$H^* = i(U-I)(U+I)^{-1} = i(I+U)^{-1}(U-I) = H.$$

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5. If A is a complex $n \times n$ matrix that is both normal and upper-triangular, then prove that A is diagonal. *Hint:* Calculate $(A^*A)_{11}$.

Solution: We prove the result by induction on n, starting with the obvious case $n = 1$. Consider then a complex upper-triangular $n \times n$ matrix A and note that

$$(A^*A)_{11} = |a_{11}|^2, \quad (AA^*)_{11} = \sum_{k=1}^{n} |a_{1k}|^2.$$

Normality $AA^* = A^*A$ then gives

$$|a_{11}|^2 = \sum_{k=1}^{n} |a_{1k}|^2 \Longrightarrow a_{1k} = 0,\ \ k > 1.$$

Thus,

$$A = \begin{pmatrix} a_{11} & 0^\top \\ 0 & A' \end{pmatrix}$$

where 0 is the $(n-1)$-dimensional column vector of zeros and A' is $(n-1) \times (n-1)$. Since it is easy to see that A' is upper-triangular and normal, the induction hypothesis that A' is diagonal implies that A is diagonal.
