Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Real Analysis

Monday, August 21, 2023

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit. However, hints are optional and solutions that don't use the hints are welcome too.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. (a) Compute $\lim_{n\to\infty} \sqrt[n]{n}$.
   (b) For any $x \in \mathbb{R}$ show that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.
   Note: If you use the fact that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, then you must prove that this power series converges for all $x \in \mathbb{R}$.

2. Prove that if $f : [a, b] \to \mathbb{R}$ is a monotonically decreasing function in the compact interval $[a, b]$, then f is Riemann integrable in $[a, b]$.

   Hint: One possible way to solve this problem is by using the following definition of Riemann integrable functions. For each uniform partition $x_i = a + i\Delta_n$, $i = 0, 1, ..., n$ with $\Delta_n = (b - a)/n$, consider the partial sums

$$U_n(f, a, b) := \sum_{i=0}^{n-1} \sup_{x\in[x_i,x_{i+1}]} f(x) \cdot \Delta_n, \quad L_n(f, a, b) := \sum_{i=0}^{n-1} \inf_{x\in[x_i,x_{i+1}]} f(x) \cdot \Delta_n$$

   and define the upper and lower Riemann integrals by

$$U(f, a, b) = \inf_n U_n(f, a, b), \quad L(f, a, b) = \sup_n L_n(f, a, b),$$

   A function is said to be Riemann integrable if $L(f, a, b) = U(f, a, b)$. It may be helpful to show that $L(f, a, b) \leq U(f, a, b)$ for all f and $\lim_{n\to\infty}[U_n(f, a, b) - L_n(f, a, b)] = 0$ for f monotone decreasing.

3. Show that the equation $z^3 + 2z + \exp(z - x - y^2) = \cos(x - y + z)$ defines implicitly a function $z = f(x, y)$ near the point $(0, 0, 0)$ where f is infinitely differentiable. Compute the differential of f at $(0, 0)$.

4. Consider a sequence of nonnegative real numbers $a_n, n \geq 0$. Show that

$$\prod_{n=0}^{\infty}(1 + a_n) \text{ converges} \quad \text{if and only if} \quad \sum_{n=0}^{\infty} a_n \text{ converges.}$$

   The following inequalities may be helpful, and can be used without proof:
   $x/2 \leq \log(1 + x) \leq x$ for $x \in [0, 2]$.

5. Denote by $\mathcal{C}[0, 1]$ the space of continuous functions $f : [0, 1] \to \mathbb{R}$ and define the subset $\mathcal{F} = \{f \in \mathcal{C}[0, 1] : \|f\| \leq 1\}$ with $\|f\| := \max_{x\in[0,1]} |f(x)|$. Show that $\mathcal{F}$ is closed and bounded in $\mathcal{C}[0, 1]$ in the norm topology. Is $\mathcal{F}$ compact or, equivalently, sequentially compact?

   Hint: The Arzelà-Ascoli theorem is one way to answer the compactness question.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Fall Session
Probability

Tuesday, August 22, 2023

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7. **No calculators of any sort are needed or permitted.**

1. In a memory game, $2n$ cards containing the numbers $1, \ldots, n$, each appearing twice, are shuffled and placed face down on the table. At each turn, you may pick two cards, turn them face up, and if they match, remove them from the table. If they do not match, they are placed face down again. The game is finished when all cards are removed.

   Suppose that instead of trying to remember the cards, you pick two cards independently and uniformly at random each turn. (That is, each of the $\binom{2n}{2}$ pairs is equally likely to be chosen.) What is the expected number of turns before you finish the game?

2. Suppose that X and Y are independent exponential random variables with rates λ and ν, respectively. The probability density function of an exponential random variable with rate λ is $\lambda e^{-\lambda x}$ for $x \geq 0$.

   Find the cumulative distribution function and probability density function of $\max\{X, Y\}$.

3. A bag has one blue and one red ball. At each step, a random ball is drawn from the bag, with all balls equally likely to be chosen. The ball is then placed back in the bag with another ball of the same color. This is repeated independently 10 times. What is the probability that 4 red balls and 6 blue balls are drawn, in any order? Fully simplify your answer.

4. For $\alpha, \beta > 0$, the Gamma(α, β) distribution is the probability distribution supported on $[0, \infty)$ with density function

$$p(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}.$$

   (Here, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$, but the definition of Γ will not be necessary for this problem; you may leave answers in terms of Γ.) Suppose that U is drawn from the Gamma(α, β) distribution, we let $\sigma = U^{-1/2}$, and then $X_1, \ldots, X_n$ are drawn i.i.d. from the distribution $N(0, \sigma^2)$. Compute the density function of U conditional on $X_1 = x_1, \ldots, X_n = x_n$.

5. At the Motor Vehicle Administration, there is an infinitely long line of people and a single clerk. The amount of time in minutes that each customer takes at the counter

is an i.i.d. random variable with mean 10 minutes and standard deviation 2 minutes. Customers are served continuously, one after the other.

Let $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ be the cumulative distribution function of a standard Gaussian random variable. Let S_n be the amount of time in minutes that it takes to serve the first n customers. Find a function f such that

$$\lim_{n\to\infty} \mathbb{P}(S_n \leq f(n)) = 0.05.$$

Your answer may involve Φ^{-1}.

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Introductory Examination–Fall Session
Linear Algebra

Wednesday, August 23, 2023

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7. **No calculators of any sort are needed or permitted.**

1. Suppose that X is a real, 4×2 matrix and

$$XX^\top = \begin{pmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & ? & ? \\ 2 & -1 & ? & ? \end{pmatrix}.$$

Fill in the missing entries marked by "?".

2. Calculate the matrix exponential e^{tA} for real t and matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

3. Prove for any real $n \times m$ matrix A that $\mathrm{Ran}(A^\top A) = \mathrm{Ran}(A^\top)$, where $\mathrm{Ran}(B)$ denotes the range of a given matrix B.

4. If U is a unitary complex $n \times n$ matrix such that $I + U$ is invertible, then prove that

$$H = i(I + U)^{-1}(U - I)$$

is Hermitian. *Hint:* Show that $H^* = i(U - I)(U + I)^{-1}$.

5. If A is a complex $n \times n$ matrix that is both normal and upper-triangular, then prove that A is diagonal. *Hint:* Calculate $(A^*A)_{11}$.