Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Real Analysis

Tuesday, January 17, 2023

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n + e^{cx}$ where n is an odd positive integer and $c > 0$.

   (a) Prove that $f(x) = 0$ for some $x \in \mathbb{R}$.

   (b) Prove that the solution to $f(x) = 0$ is unique.

Solution:

(a) Since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$ and f is continuous, the existence of a root is a consequence of the intermediate value theorem.

(b) f is strictly increasing so uniqueness is immediate.

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2. Define a sequence

$$x_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \log n$$

for $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} x_n$ exists and is finite.

Hint: Show that the sequence (x_n) is nonnegative by comparing the sum in x_n to a sum of integrals. Then show that the sequence (x_n) is monotone.

Solution: We have

$$\log n = \int_{t=1}^{n} \frac{1}{t} dt = \sum_{k=1}^{n-1} \int_{t=k}^{k+1} \frac{1}{t} dt.$$

so

$$x_n = \left(\sum_{k=1}^{n-1} \frac{1}{k} - \int_{t=k}^{k+1} \frac{1}{t} dt\right) + \frac{1}{n}.$$

Since the integrand in $\int_{t=k}^{k+1} \frac{1}{t} dt$ is bounded above by $1/k$ and we are integrating over an interval of length 1, this integral is bounded above by $1/k$ and we obtain

$$x_n \geq 1/n.$$

Now

$$x_{n+1} - x_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) = \frac{1}{n+1} - \int_{t=1}^{1+\frac{1}{n}} \frac{1}{t} dt.$$

The integrand is bounded below by $\frac{1}{1+\frac{1}{n}}$ and the length of the interval of integration is $1/n$ so

$$x_{n+1} - x_n \leq \frac{1}{n+1} - \frac{1}{n} \frac{1}{1+\frac{1}{n}} = 0.$$

Since the sequence is bounded below by zero and monotone nonincreasing, it must converge.

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3. (a) Show there exists $\epsilon > 0$ such that $\log(1-x) \geq -2x$ for $x \in [0, \epsilon)$.

   (b) Define $p_n = \prod_{i=1}^{n} \frac{2^i - 1}{2^i}$ for $n = 1, 2, \ldots$. Use the result in (a) to show that the limit $\lim_{n\to\infty} p_n$ exists and is positive.

Solution: (a) Observe that the inequality holds for $x = 0$. Define $f(x) = \frac{\log(1-x)}{x}$ for $x \in (0, 1)$. Using L'hopital's rule we see that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} -\frac{1}{1-x} = -1$$

so we can define $f(0) = -1$ and f is continuous in $[0, 1)$. Since $f(0) = -1 > -2$, by continuity there exists $\epsilon > 0$ such that $f(x) > -2$ for $x \in [0, \epsilon)$ and we conclude that $\log(1-x) \geq -2x$ for $x \in [0, \epsilon)$.

Alternative solutions to (a): One can employ the mean value theorem to the function $g(x) := \log(1-x)$. Since $g'(x) = -1/(1-x)$ for $x \in [0, 1)$, for each such x we have

$$\frac{\log(1-x)}{x} = \frac{\log(1-x) - 0}{x - 0} = -\frac{1}{1-\xi}$$

for some $\xi \in (0, x)$. If $x \in (0, 1/2]$, then $\xi \in (0, 1/2)$, and so

$$\frac{\log(1-x)}{x} = -\frac{1}{1-\xi} > -2.$$

So with $\epsilon := 1/2$ we have the desired inequality for $x \in (0, \epsilon)$, and the desired inequality holds with equality when $x = 0$.

Here's an even easier solution. Let $h(x) := \log(1-x) + 2x$ for $x \in [0, 1)$. Then $h(0) = 0$ and $h'(x) = 2 - (1-x)^{-1}$, which is nonnegative for $x \in [0, 1/2]$. Hence $h(x) \geq h(0) = 0$ for $x \in [0, 1/2]$.

(b) Note that the terms in $\log p_n = \sum_{i=1}^n \log(1 - \frac{1}{2^i})$ are all negative, so $\log p_n$ is monotone decreasing and hence converges to some limit (possibly $-\infty$). It suffices therefore to find a finite lower bound for the sum.

From (a), let $\epsilon > 0$ be such that

$$\log(1-x) \geq -2x$$

for $x \in [0, \epsilon)$. Take I such that $1/2^I < \epsilon$. Then for $n \geq I$ we have

$$\begin{aligned}
\log p_n = \sum_{i=1}^{n} \log\left(1 - \frac{1}{2^i}\right) &= \sum_{i=1}^{I-1} \log\left(1 - \frac{1}{2^i}\right) + \sum_{i=I}^{n} \log\left(1 - \frac{1}{2^i}\right) \\
&\geq \sum_{i=1}^{I-1} \log\left(1 - \frac{1}{2^i}\right) - 2\sum_{i=I}^{n} \frac{1}{2^i} \\
&\geq \sum_{i=1}^{I-1} \log\left(1 - \frac{1}{2^i}\right) - 2\sum_{i=I}^{\infty} \frac{1}{2^i} \\
&= \sum_{i=1}^{I-1} \log\left(1 - \frac{1}{2^i}\right) - 2^{2-I}.
\end{aligned}$$

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4. A sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \ldots.$ is said to be *uniformly equicontinuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all n whenever $|x - y| < \delta$.

   Suppose the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \ldots$ is uniformly equicontinuous, and suppose the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$. Show that the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous.

Solution: Fix $\epsilon > 0$. By uniform equicontinuity, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon/3$ for all n whenever $|x - y| < \delta$. Fix x, y with $|x - y| < \delta$. Since $f_n(x) \to f(x)$ there exists N_1 such that $|f_n(x) - f(x)| < \epsilon/3$ for all $n \geq N_1$, and since $f_n(y) \to f(y)$ there exists N_2 such that $|f_n(y) - f(y)| < \epsilon/3$ for all $n \geq N_2$. For $n \geq \max\{N_1, N_2\}$ the triangle inequality gives

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

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5. Suppose $U \subset \mathbb{R}$ is open. Show that

$$U \subseteq \overline{U \cap \mathbb{Q}}.$$

Here $\overline{V}$ denotes the closure of a set $V \subseteq R$, and $\mathbb{Q}$ denotes the set of rational numbers.

Solution: Given $u \in U$, it suffices to show every open neighborhood V of u contains a point in $U \cap \mathbb{Q}$. Let V be an open neighborhood of u. Then $V \cap U$ is also an open neighborhood of u, and since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have $(V \cap U) \cap \mathbb{Q} \neq \emptyset$. Thus

$$V \cap (U \cap \mathbb{Q}) = (V \cap U) \cap \mathbb{Q} \neq \emptyset.$$

Alternative solution: Fix $u \in U$. Since U is open, there exists N such that $B(u, 1/N) \subseteq U$. We can define a sequence $(q_n)_{n=N}^{\infty} \subseteq U \cap \mathbb{Q}$ as follows. For any $n \geq N$, we have $B(u, 1/n) \subseteq B(u, 1/N) \subseteq U$. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and $B(u, 1/N)$ is open, there exists $q_n \in \mathbb{Q} \cap B(u, 1/n)$, and as a consequence $q_n \in U$. Since $|q_n - u| < 1/n$ we conclude that $\lim_{n\to\infty} q_n = u$.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Probability

Wednesday, January 18, 2022

Instructions: Read carefully!

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7. **No calculators of any sort are needed or permitted.**

1. In a box there are 99 fair coins and 1 double-headed coin (i.e., it always comes up heads). Suppose that I randomly draw a coin, flip it n times, and it comes up heads every time. What is the smallest n such that, given this information, the probability that the coin is double-headed is at least $\frac{9}{10}$?

Solution: Let H_n denote the event that the n tosses are all heads, and A denote the event that I drew the double-headed coin. By Bayes's Rule,

$$\begin{aligned} P(A|H_n) &= \frac{P(A \cap H_n)}{P(H_n)} = \frac{P(A)P(H_n|A)}{P(A)P(H_n|A) + P(A^c)P(H_n|A^c)} \\ &= \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n}. \end{aligned}$$

We solve

$$\begin{aligned} P(A|H_n) &\geq \frac{9}{10} \\ \iff \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \frac{1}{2}^n} &\geq \frac{9}{10} \\ \iff \frac{1}{100} &\geq 9 \cdot \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n \\ \iff 2^n &\geq 9 \cdot 99 \\ \iff n &\geq 10, \end{aligned}$$

keeping in mind that n must be a natural number. The smallest n is $\boxed{10}$.

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2. In a group of n people, suppose that each person has equal probability $\frac{1}{365}$ of having their birthday on any day of the year (assume Feb 29th is not possible), and the birthdays are independent. We say a birthday is *unique* if exactly one person in the group has that birthday. What is the expected number of unique birthdays, as a function of n?

Solution: Number the days of the year from 1 to 365 and the people from 1 to n. Let X_k be the indicator random variable that day k is a unique birthday, and U be the total number of unique birthdays. Note that $U = \sum_{k=1}^{365} X_k$. By linearity of expectation,

$$\mathbb{E}U = \sum_{k=1}^{365} \mathbb{E}X_k = \sum_{k=1}^{365} P(k \text{ is a unique birthday}).$$

Now

$$\begin{aligned}
&P(k \text{ is a unique birthday}) \\
&= \sum_{m=1}^{n} P(\text{person } m \text{ has birthday } k, \text{ and everyone else has a birthday} \neq k) \\
&= \sum_{m=1}^{n} \frac{1}{365}\left(\frac{364}{365}\right)^{n-1} = \frac{n}{365}\left(\frac{364}{365}\right)^{n-1}.
\end{aligned}$$

Thus

$$\mathbb{E}U = 365 \cdot \frac{n}{365}\left(\frac{364}{365}\right)^{n-1} = \boxed{n\left(\frac{364}{365}\right)^{n-1}}.$$

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3. 6 ducks and 6 geese randomly sit down in a circle, with each permutation equally likely. Compute the probability that there are at least 4 ducks in a row somewhere in the circle.

Solution: Label the positions clockwise from 1 to 12, and abbreviate ducks as D and geese as G. We note that there are at least 4 ducks in a row somewhere in the circle if and only if, starting from some location i and reading clockwise, the birds at those locations are GDDDD. Letting A_i denote this event, we wish to find the probability of $\bigcup_{i=1}^{12} A_i$.

Note

$$P(A_i) = \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8},$$

as we may as well assume we are drawing ducks and geese randomly starting at the ith location, and are sampling the ducks and geese without replacement. The A_i are disjoint, as it is impossible to have two sequences of at least 4 ducks. Hence

$$P\left(\bigcup_{i=1}^{12} A_i\right) = \sum_{i=1}^{12} P(A_i) = 12 \cdot \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8} = \boxed{\frac{3}{11}}.$$

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4. Let X_1, X_2, and X_3 be independent random variables drawn from the exponential distribution with mean 1. Compute the probability density function of X_2 conditional on the event $X_1 \leq X_2 \leq X_3$.

   (Equivalently, let $Y_1 \leq Y_2 \leq Y_3$ be X_1, X_2, X_3 sorted in increasing order. Compute the probability density function of Y_2.)

Solution:

Solution 1. The joint density function of (X_1, X_2, X_3) is $e^{-(x_1+x_2+x_3)}$, for $x_1, x_2, x_3 \geq 0$. The density function of X_2 conditional on $X_1 \leq X_2 \leq X_3$ is (for $x_2 \geq 0$)

$$\frac{\int_{x_1,x_3:0\leq x_1\leq x_2\leq x_3} e^{-(x_1+x_2+x_3)}\,dx_1\,dx_3}{\int_{0\leq x_1\leq x_2\leq x_3} e^{-(x_1+x_2+x_3)}\,dx_1\,dx_2\,dx_3}.$$

Let $f(x_2)$ denote the numerator and Z denote the denominator. The numerator is

$$\begin{aligned} f(x_2) &= e^{-x_2}\int_{x_2}^{\infty} e^{-x_3}\int_0^{x_2} e^{-x_1}\,dx_1\,dx_3 \\ &= e^{-x_2}e^{-x_2}(1-e^{-x_2}) = e^{-2x_2}(1-e^{-x_2}). \end{aligned}$$

The denominator is

$$Z = \int_0^{\infty} f(x_2)\,dx_2 = \int_0^{\infty}\left(e^{-2x_2} - e^{-3x_2}\right)\,dx_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The conditional probability density function is then

$$\frac{f(x)}{Z} = \boxed{6e^{-2x}(1-e^{-x})}, \quad x \geq 0.$$

Solution 2. Let $F(x)$ denote the cumulative distribution function for the exponential distribution with mean 1; note $F(x) = 1 - e^{-x}$ for $x \geq 0$. By symmetry, conditional on any ordering $X_{\pi(1)} \leq X_{\pi(2)} \leq X_{\pi(3)}$, the distribution of $X_{\pi(2)}$ is the same. Hence, if we let $Y_1 \leq Y_2 \leq Y_3$ be X_1, X_2, X_3 sorted in increasing order, then an equivalent problem is to find the density function of Y_2.

Note that when F is a continuous cdf, the inverse cdf F^{-1} transforms a uniform random variable to a random variable with cdf F. Because F^{-1} is monotonic, (Y_1, Y_2, Y_3) has the same distribution as $(F^{-1}(Z_1), F^{-1}(Z_2), F^{-1}(Z_3))$, where (Z_1, Z_2, Z_3) are uniform order statistics. We know that Z_2 has a beta distribution with $\alpha = 2$ and $\beta = 2$, which has density $g(x) := \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!}x^{\alpha-1}(1-x)^{\beta-1} = 6x(1-x)$ for $0 \leq x \leq 1$.

By the change-of-variables formula, $F^{-1}(Z_2)$ then has density function

$$g(F(x))F'(x) = 6F(x)(1-F(x))e^{-x} = 6e^{-2x}(1-e^{-x}), \quad x \geq 0.$$

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5. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ whose entries are independently drawn from a standard normal distribution $X_k \sim N(0,1)$. For $x \in \mathbb{R}^n$, let $\|x\| =$

$\sqrt{x_1^2 + \cdots + x_n^2}$. For any $\epsilon > 0$, show that there exists a constant C_ϵ such that for all large enough n,

$$P(\|X\| - \sqrt{n} > C_\epsilon) < \epsilon.$$

Solution:

Solution 1. Let $S_n = X_1^2 + \cdots + X_n^2$; note $\|X\|^2 = S_n$. Then for any $C > 0$,

$$\begin{aligned} P(\|X\| - \sqrt{n} > C) &= P(\|X\| > \sqrt{n} + C) \\ &= P(\|X\|^2 > n + 2C\sqrt{n} + C^2) \\ &\leq P(S_n > n + 2C\sqrt{n}) \\ &= P\left(\frac{S_n - n}{\sqrt{n}} > 2C\right). \end{aligned}$$

Now, S_n is a sum of iid copies of X_1^2, and X_1^2 has variance $\sigma^2 := \mathrm{Var}(X_1^2) = \mathbb{E}X_1^4 - (\mathbb{E}X_1^2)^2 = 2 < \infty$. (The exact value is not important.) Thus by the central limit theorem, $\frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z$, where $Z \sim N(0,1)$ is standard normal. Let Φ be the cdf of the normal distribution and $C_\epsilon = \frac{\sigma}{2}\Phi^{-1}(1 - \frac{\epsilon}{2})$. Then as $n \to \infty$,

$$\begin{aligned} P\left(\frac{S_n - n}{\sqrt{n}} > 2C_\epsilon\right) &= P\left(\frac{S_n - n}{\sqrt{n}} > \sigma\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) \\ &\to P_{Z\sim N(0,1)}\left(\sigma Z > \sigma\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) = \frac{\epsilon}{2}, \end{aligned}$$

using convergence in distribution. For n large enough, we will still have $P\left(\frac{S_n - n}{\sqrt{n}} > 2C_\epsilon\right) < \epsilon$, which gives the required bound.

Solution 2. Defining S_n as in Solution 1, we have that

$$\|X\| - \sqrt{n} = \sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{n}} \frac{\sqrt{n}}{\sqrt{S_n} + \sqrt{n}}.$$

As before, by the central limit theorem, $\frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z$, $Z \sim N(0,1)$. By the weak law of large numbers, $\frac{S_n}{n} \to \mathbb{E}X_1^2 = 1$ in probability, so $\frac{\sqrt{n}}{\sqrt{S_n} + \sqrt{n}} = \frac{1}{\sqrt{S_n/n} + 1} \to \frac{1}{2}$ in probability. Hence $\sqrt{S_n} - \sqrt{n} \Rightarrow \frac{1}{2}\sigma Z$. Setting $C_\epsilon = \frac{\sigma}{2}\Phi^{-1}(1 - \frac{\epsilon}{2})$ we have as $n \to \infty$,

$$P(\sqrt{S_n} - \sqrt{n} > C_\epsilon) \to P_{Z\sim N(0,1)}\left(\frac{\sigma}{2}Z > \frac{\sigma}{2}\Phi^{-1}\left(1 - \frac{\epsilon}{2}\right)\right) = \frac{\epsilon}{2},$$

and finish as before.

Applied Mathematics and Statistics
Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Linear Algebra

Thursday, January 19, 2023

Instructions: Read carefully!

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7. **No calculators of any sort are needed or permitted.**

1. Let $P_1, P_2 \in \mathbb{R}^{n \times n}$ be projection matrices. Show that if $P_1 - P_2$ is a projection matrix, then $\text{range}(P_2) \subseteq \text{range}(P_1)$.

2. Find the rank of
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

3. Consider the matrix A below:
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$
If A is diagonalizable, find a diagonalizing matrix S such that $S^{-1}AS$ is diagonal. Otherwise, show that A is not diagonalizable.

4. Is there an invertible matrix $A \in \mathbb{R}^{n \times n}$ for any value of $n \geq 2$ for which every $(n-1) \times (n-1)$ minor (not necessarily principal) is equal to 1? Prove that no such matrix can exist, or find a matrix with this property.

5. Consider the real vector space V of polynomials with real coefficients having degree at most d, where $d \geq 2$ is an integer. Show that the subspaces S_1, S_2 satisfy $\dim(S_1 \cap S_2) \geq (d-1)/2$, where these are defined as:
   - $S_1 = \{p \in V : p(1) = 0\}$.
   - $S_2 = \{p \in V : p \text{ is even, i.e., } p(x) = p(-x) \text{ for all } x\}$.