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7. No calculators of any sort are needed or permitted.
1. Let the function \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^n + e^{cx} \) where \( n \) is an odd positive integer and \( c > 0 \).

   (a) Prove that \( f(x) = 0 \) for some \( x \in \mathbb{R} \).
   (b) Prove that the solution to \( f(x) = 0 \) is unique.

**Solution:**

(a) Since \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to \infty} f(x) = \infty \) and \( f \) is continuous, the existence of a root is a consequence of the intermediate value theorem.

(b) \( f \) is strictly increasing so uniqueness is immediate.

2. Define a sequence

\[ x_n = \left( \sum_{k=1}^{n} \frac{1}{k} \right) - \log n \]

for \( n = 1, 2, \ldots \). Prove that \( \lim_{n \to \infty} x_n \) exists and is finite.

Hint: Show that the sequence \((x_n)\) is nonnegative by comparing the sum in \( x_n \) to a sum of integrals. Then show that the sequence \((x_n)\) is monotone.

**Solution:** We have

\[ \log n = \int_{t=1}^{n} \frac{1}{t} dt = \sum_{k=1}^{n-1} \int_{t=k}^{k+1} \frac{1}{t} dt. \]

so

\[ x_n = \left( \sum_{k=1}^{n-1} \frac{1}{k} - \int_{t=k}^{k+1} \frac{1}{t} dt \right) + \frac{1}{n}. \]

Since the integrand in \( \int_{t=k}^{k+1} \frac{1}{t} dt \) is bounded above by \( 1/k \) and we are integrating over an interval of length 1, this integral is bounded above by \( 1/k \) and we obtain

\[ x_n \geq 1/n. \]

Now

\[ x_{n+1} - x_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log \left( \frac{n+1}{n} \right) \]
\[
\frac{1}{n+1} - \log \left( 1 + \frac{1}{n} \right) = \frac{1}{n+1} - \int_{t=1}^{1+\frac{1}{n}} \frac{1}{t} \, dt.
\]

The integrand is bounded below by \( \frac{1}{1+\frac{1}{n}} \) and the length of the interval of integration is \( 1/n \) so

\[
x_{n+1} - x_n \leq \frac{1}{n+1} - \frac{1}{n} \frac{1}{1+\frac{1}{n}} = 0.
\]

Since the sequence is bounded below by zero and monotone nonincreasing, it must converge.

3. (a) Show there exists \( \epsilon > 0 \) such that \( \log(1 - x) \geq -2x \) for \( x \in [0, \epsilon) \).

(b) Define \( p_n = \prod_{i=1}^{n} \frac{2i-1}{2i} \) for \( n = 1, 2, \ldots \). Use the result in (a) to show that the limit \( \lim_{n \to \infty} p_n \) exists and is positive.

**Solution:** (a) Observe that the inequality holds for \( x = 0 \). Define \( f(x) = \frac{\log(1-x)}{x} \) for \( x \in (0, 1) \). Using L’hopital’s rule we see that

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} -\frac{1}{1-x} = -1
\]

so we can define \( f(0) = -1 \) and \( f \) is continuous in \([0, 1) \). Since \( f(0) = -1 > -2 \), by continuity there exists \( \epsilon > 0 \) such that \( f(x) > -2 \) for \( x \in [0, \epsilon) \) and we conclude that \( \log(1 - x) \geq -2x \) for \( x \in [0, \epsilon) \).

Alternative solutions to (a): One can employ the mean value theorem to the function \( g(x) := \log(1-x) \). Since \( g'(x) = -1/(1-x) \) for \( x \in [0, 1) \), for each such \( x \) we have

\[
\frac{\log(1-x)}{x} = \frac{\log(1-x) - 0}{x - 0} = -\frac{1}{1-x} \xi
\]

for some \( \xi \in (0, x) \). If \( x \in (0, 1/2] \), then \( \xi \in (0, 1/2) \), and so

\[
\frac{\log(1-x)}{x} = -\frac{1}{1-x} \xi > -2.
\]

So with \( \epsilon := 1/2 \) we have the desired inequality for \( x \in (0, \epsilon) \), and the desired inequality holds with equality when \( x = 0 \).

Here’s an even easier solution. Let \( h(x) := \log(1-x) + 2x \) for \( x \in [0, 1) \). Then \( h(0) = 0 \) and \( h'(x) = 2 - (1-x)^{-1} \), which is nonnegative for \( x \in [0, 1/2] \). Hence \( h(x) \geq h(0) = 0 \) for \( x \in [0, 1/2] \).
(b) Note that the terms in \( \log p_n = \sum_{i=1}^{n} \log(1 - \frac{1}{2^i}) \) are all negative, so \( \log p_n \) is monotone decreasing and hence converges to some limit (possibly \(-\infty\)). It suffices therefore to find a finite lower bound for the sum.

From (a), let \( \epsilon > 0 \) be such that
\[
\log(1 - x) \geq -2x
\]
for \( x \in [0, \epsilon) \). Take \( I \) such that \( 1/2^I < \epsilon \). Then for \( n \geq I \) we have
\[
\log p_n = \sum_{i=1}^{n} \log \left( 1 - \frac{1}{2^i} \right) = \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) + \sum_{i=I}^{n} \log \left( 1 - \frac{1}{2^i} \right)
\]
\[
\geq \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2 \sum_{i=I}^{n} \frac{1}{2^i}
\]
\[
\geq \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2 \sum_{i=I}^{\infty} \frac{1}{2^i}
\]
\[
= \sum_{i=1}^{I-1} \log \left( 1 - \frac{1}{2^i} \right) - 2^{2-I}.
\]

4. A sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \) for \( n = 1, 2, \ldots \) is said to be **uniformly equicontinuous** if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \epsilon \) for all \( n \) whenever \( |x - y| < \delta \).

Suppose the sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \) for \( n = 1, 2, \ldots \) is uniformly equicontinuous, and suppose the pointwise limit \( \lim_{n \to \infty} f_n(x) \) exists for all \( x \in \mathbb{R} \). Show that the function defined by \( f(x) = \lim_{n \to \infty} f_n(x) \) is continuous.

**Solution:** Fix \( \epsilon > 0 \). By uniform equicontinuity, there exists \( \delta > 0 \) such that \( |f_n(x) - f_n(y)| < \epsilon/3 \) for all \( n \) whenever \( |x - y| < \delta \). Fix \( x, y \) with \( |x - y| < \delta \). Since \( f_n(x) \to f(x) \) there exists \( N_1 \) such that \( |f_n(x) - f(x)| < \epsilon/3 \) for all \( n \geq N_1 \), and since \( f_n(y) \to f(y) \) there exists \( N_2 \) such that \( |f_n(y) - f(y)| < \epsilon/3 \) for all \( n \geq N_2 \). For \( n \geq \max\{N_1, N_2\} \) the triangle inequality gives
\[
|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(x) - f(x)| + |f_n(y) - f_n(x)|
\]
\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]
5. Suppose $U \subset \mathbb{R}$ is open. Show that

$$U \subseteq U \cap \mathbb{Q}.$$  

Here $\overline{V}$ denotes the closure of a set $V \subseteq \mathbb{R}$, and $\mathbb{Q}$ denotes the set of rational numbers.

**Solution:** Given $u \in U$, it suffices to show every open neighborhood $V$ of $u$ contains a point in $U \cap \mathbb{Q}$. Let $V$ be an open neighborhood of $u$. Then $V \cap U$ is also an open neighborhood of $u$, and since $\mathbb{Q}$ is dense in $\mathbb{R}$ we have $(V \cap U) \cap \mathbb{Q} \neq \emptyset$. Thus

$$V \cap (U \cap \mathbb{Q}) = (V \cap U) \cap \mathbb{Q} \neq \emptyset.$$  

Alternative solution: Fix $u \in U$. Since $U$ is open, there exists $N$ such that $B(u, 1/N) \subseteq U$. We can define a sequence $(q_n)_{n=N}^{\infty} \subseteq U \cap \mathbb{Q}$ as follows. For any $n \geq N$, we have $B(u, 1/n) \subseteq B(u, 1/N) \subseteq U$. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and $B(u, 1/N)$ is open, there exists $q_n \in \mathbb{Q} \cap B(u, 1/n)$, and as a consequence $q_n \in U$. Since $|q_n - u| < 1/n$ we conclude that $\lim_{n \to \infty} q_n = u$. 

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7. **No calculators of any sort are needed or permitted.**
1. In a box there are 99 fair coins and 1 double-headed coin (i.e., it always comes up heads). Suppose that I randomly draw a coin, flip it \( n \) times, and it comes up heads every time. What is the smallest \( n \) such that, given this information, the probability that the coin is double-headed is at least \( \frac{9}{10} \)?

**Solution:** Let \( H_n \) denote the event that the \( n \) tosses are all heads, and \( A \) denote the event that I drew the double-headed coin. By Bayes’s Rule,

\[
P(A|H_n) = \frac{P(A \cap H_n)}{P(H_n)} = \frac{P(A)P(H_n|A)}{P(A)P(H_n|A) + P(A^c)P(H_n|A^c)}
\]

\[
= \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n}.
\]

We solve

\[
P(A|H_n) \geq \frac{9}{10}
\]

\[
\iff \frac{\frac{1}{100} \cdot 1}{\frac{1}{100} \cdot 1 + \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n} \geq \frac{9}{10}
\]

\[
\iff \frac{1}{100} \geq 9 \cdot \frac{99}{100} \cdot \left(\frac{1}{2}\right)^n
\]

\[
\iff 2^n \geq 9 \cdot 99
\]

\[
\iff n \geq 10,
\]

keeping in mind that \( n \) must be a natural number. The smallest \( n \) is 10.

2. In a group of \( n \) people, suppose that each person has equal probability \( \frac{1}{365} \) of having their birthday on any day of the year (assume Feb 29th is not possible), and the birthdays are independent. We say a birthday is unique if exactly one person in the group has that birthday. What is the expected number of unique birthdays, as a function of \( n \)?

**Solution:** Number the days of the year from 1 to 365 and the people from 1 to \( n \). Let \( X_k \) be the indicator random variable that day \( k \) is a unique birthday, and \( U \) be the total number of unique birthdays. Note that \( U = \sum_{k=1}^{365} X_k \). By linearity of expectation,

\[
\mathbb{E}U = \sum_{k=1}^{365} \mathbb{E}X_k = \sum_{k=1}^{365} P(k \text{ is a unique birthday}).
\]
Now

\[ P(k \text{ is a unique birthday}) = \sum_{m=1}^{n} P(\text{person } m \text{ has birthday } k, \text{ and everyone else has a birthday } \neq k) \]

\[ = \sum_{m=1}^{n} \frac{1}{365} \left( \frac{364}{365} \right)^{n-1} = \frac{n}{365} \left( \frac{364}{365} \right)^{n-1}. \]

Thus

\[ EU = 365 \cdot \frac{n}{365} \left( \frac{364}{365} \right)^{n-1} = n \left( \frac{364}{365} \right)^{n-1}. \]

3. 6 ducks and 6 geese randomly sit down in a circle, with each permutation equally likely. Compute the probability that there are at least 4 ducks in a row somewhere in the circle.

Solution: Label the positions clockwise from 1 to 12, and abbreviate ducks as D and geese as G. We note that there are at least 4 ducks in a row somewhere in the circle if and only if, starting from some location \( i \) and reading clockwise, the birds at those locations are GDDDD. Letting \( A_i \) denote this event, we wish to find the probability of \( \bigcup_{i=1}^{12} A_i \).

Note

\[ P(A_i) = \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}, \]

as we may as well assume we are drawing ducks and geese randomly starting at the \( i \)th location, and are sampling the ducks and geese without replacement. The \( A_i \) are disjoint, as it is impossible to have two sequences of at least 4 ducks. Hence

\[ P\left( \bigcup_{i=1}^{12} A_i \right) = \sum_{i=1}^{12} P(A_i) = 12 \cdot \frac{6 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8} = \frac{3}{11}. \]

4. Let \( X_1, X_2, \) and \( X_3 \) be independent random variables drawn from the exponential distribution with mean 1. Compute the probability density function of \( X_2 \) conditional on the event \( X_1 \leq X_2 \leq X_3 \).

(Equivalently, let \( Y_1 \leq Y_2 \leq Y_3 \) be \( X_1, X_2, X_3 \) sorted in increasing order. Compute the probability density function of \( Y_2 \).)
Solution:

Solution 1. The joint density function of \((X_1, X_2, X_3)\) is \(e^{-(x_1 + x_2 + x_3)}\), for \(x_1, x_2, x_3 \geq 0\). The density function of \(X_2\) conditional on \(X_1 \leq X_2 \leq X_3\) is (for \(x_2 \geq 0\))

\[
\frac{\int_{x_1, x_3: 0 \leq x_1 \leq x_2 \leq x_3} e^{-(x_1 + x_2 + x_3)} \, dx_1 \, dx_3}{\int_{0 \leq x_1 \leq x_2 \leq x_3} e^{-(x_1 + x_2 + x_3)} \, dx_1 \, dx_2 \, dx_3}.
\]

Let \(f(x_2)\) denote the numerator and \(Z\) denote the denominator. The numerator is

\[
f(x_2) = e^{-x_2} \int_{x_2}^{\infty} e^{-x_3} \, dx_3 \int_{0}^{x_2} e^{-x_1} \, dx_1 \, dx_3
= e^{-x_2} e^{-x_2} (1 - e^{-x_2}) = e^{-2x_2} (1 - e^{-x_2}).
\]

The denominator is

\[
Z = \int_{0}^{\infty} f(x_2) \, dx_2 = \int_{0}^{\infty} (e^{-2x_2} - e^{-3x_2}) \, dx_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\]

The conditional probability density function is then

\[
\frac{f(x)}{Z} = \frac{6e^{-2x}(1 - e^{-x})}{x \geq 0}.
\]

Solution 2. Let \(F(x)\) denote the cumulative distribution function for the exponential distribution with mean 1; note \(F(x) = 1 - e^{-x}\) for \(x \geq 0\). By symmetry, conditional on any ordering \(X_{\pi(1)} \leq X_{\pi(2)} \leq X_{\pi(3)}\), the distribution of \(X_{\pi(2)}\) is the same. Hence, if we let \(Y_1 \leq Y_2 \leq Y_3\) be \(X_1, X_2, X_3\) sorted in increasing order, then an equivalent problem is to find the density function of \(Y_2\).

Note that when \(F\) is a continuous cdf, the inverse cdf \(F^{-1}\) transforms a uniform random variable to a random variable with cdf \(F\). Because \(F^{-1}\) is monotonic, \((Y_1, Y_2, Y_3)\) has the same distribution as \((F^{-1}(Z_1), F^{-1}(Z_2), F^{-1}(Z_3))\), where \((Z_1, Z_2, Z_3)\) are uniform order statistics. We know that \(Z_2\) has a beta distribution with \(\alpha = 2\) and \(\beta = 2\), which has density \(g(x) := \frac{(\alpha + \beta - 1)!}{(\alpha - 1)! (\beta - 1)!} x^{\alpha - 1} (1 - x)^{\beta - 1} = 6x(1 - x)\) for \(0 \leq x \leq 1\).

By the change-of-variables formula, \(F^{-1}(Z_2)\) then has density function

\[
g(F(x)) F'(x) = 6F(x)(1 - F(x)) e^{-x} = 6e^{-2x}(1 - e^{-x}), \quad x \geq 0.
\]

5. Let \(X = (X_1, \ldots, X_n)\) be a random vector in \(\mathbb{R}^n\) whose entries are independently drawn from a standard normal distribution \(X_k \sim N(0, 1)\). For \(x \in \mathbb{R}^n\), let \(\|x\| = \sum_{i=1}^{n} x_i^2\).
\[ \sqrt{x_1^2 + \cdots + x_n^2}. \] For any \( \epsilon > 0 \), show that there exists a constant \( C_\epsilon \) such that for all large enough \( n \),

\[ P(||X|| - \sqrt{n} > C_\epsilon) < \epsilon. \]

**Solution:**

**Solution 1.** Let \( S_n = X_1^2 + \cdots + X_n^2 \); note \( ||X||^2 = S_n \). Then for any \( C > 0 \),

\[
P(||X|| - \sqrt{n} > C) = P(||X|| > \sqrt{n} + C) \\
= P(||X||^2 > n + 2C\sqrt{n} + C^2) \\
\leq P(S_n > n + 2C\sqrt{n}) \\
= P \left( \frac{S_n - n}{\sqrt{n}} > 2C \right).
\]

Now, \( S_n \) is a sum of iid copies of \( X_1^2 \), and \( X_1^2 \) has variance \( \sigma^2 := \text{Var}(X_1^2) = \mathbb{E}X_1^4 - (\mathbb{E}X_1^2)^2 = 2 < \infty \). (The exact value is not important.) Thus by the central limit theorem, \( \frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z \), where \( Z \sim N(0, 1) \) is standard normal. Let \( \Phi \) be the cdf of the normal distribution and \( C_\epsilon = \frac{\sigma}{2} \Phi^{-1}(1 - \frac{\epsilon}{2}) \). Then as \( n \to \infty \),

\[
P \left( \frac{S_n - n}{\sqrt{n}} > 2C_\epsilon \right) = P \left( \frac{S_n - n}{\sqrt{n}} > \sigma \Phi^{-1} \left( 1 - \frac{\epsilon}{2} \right) \right) \\
\to P_{Z \sim N(0, 1)} \left( \sigma Z > \sigma \Phi^{-1} \left( 1 - \frac{\epsilon}{2} \right) \right) = \frac{\epsilon}{2},
\]

using convergence in distribution. For \( n \) large enough, we will still have \( P \left( \frac{S_n - n}{\sqrt{n}} > 2C_\epsilon \right) < \epsilon \), which gives the required bound.

**Solution 2.** Defining \( S_n \) as in Solution 1, we have that

\[ ||X|| - \sqrt{n} = \sqrt{S_n} - \sqrt{n} = \frac{S_n - n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{S_n + \sqrt{n}}}. \]

As before, by the central limit theorem, \( \frac{S_n - n}{\sqrt{n}} \Rightarrow \sigma Z, Z \sim N(0, 1) \). By the weak law of large numbers, \( \frac{S_n}{n} \to \mathbb{E}X_1^2 = 1 \) in probability, so \( \frac{\sqrt{n}}{\sqrt{S_n + \sqrt{n}}} = \frac{1}{\sqrt{S_n/n + 1}} \to \frac{1}{2} \) in probability. Hence \( \sqrt{S_n} - \sqrt{n} \Rightarrow \frac{1}{2}\sigma Z \). Setting \( C_\epsilon = \frac{\sigma}{2} \Phi^{-1}(1 - \frac{\epsilon}{2}) \) we have as \( n \to \infty \),

\[
P(\sqrt{S_n} - \sqrt{n} > C_\epsilon) \to P_{Z \sim N(0, 1)} \left( \frac{\sigma}{2} Z > \frac{\sigma}{2} \Phi^{-1} \left( 1 - \frac{\epsilon}{2} \right) \right) = \frac{\epsilon}{2},
\]

and finish as before.
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1. Let $P_1, P_2 \in \mathbb{R}^{n \times n}$ be projection matrices. Show that if $P_1 - P_2$ is a projection matrix, then $\text{range}(P_2) \subseteq \text{range}(P_1)$.

2. Find the rank of

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

3. Consider the matrix $A$ below:

$$
A = \begin{bmatrix}
2 & 3 \\
0 & 1
\end{bmatrix}
$$

If $A$ is diagonalizable, find a diagonalizing matrix $S$ such that $S^{-1}AS$ is diagonal. Otherwise, show that $A$ is not diagonalizable.

4. Is there an invertible matrix $A \in \mathbb{R}^{n \times n}$ for any value of $n \geq 2$ for which every $(n - 1) \times (n - 1)$ minor (not necessarily principal) is equal to 1? Prove that no such matrix can exist, or find a matrix with this property.

5. Consider the real vector space $V$ of polynomials with real coefficients having degree at most $d$, where $d \geq 2$ is an integer. Show that the subspaces $S_1, S_2$ satisfy

$$
\dim(S_1 \cap S_2) \geq (d - 1)/2,
$$

where these are defined as:

- $S_1 = \{p \in V : p(1) = 0\}$.
- $S_2 = \{p \in V : p$ is even, i.e., $p(x) = p(-x)$ for all $x\}$. 