Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Real Analysis

Tuesday, January 17, 2023

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n + e^{cx}$ where n is an odd positive integer and $c > 0$.

   (a) Prove that $f(x) = 0$ for some $x \in \mathbb{R}$.

   (b) Prove that the solution to $f(x) = 0$ is unique.

2. Define a sequence

$$x_n = \left(\sum_{k=1}^{n} \frac{1}{k} \right) - \log n$$

   for $n = 1, 2, \ldots$. Prove that $\lim_{n\to\infty} x_n$ exists and is finite.

   Hint: Show that the sequence (x_n) is nonnegative by comparing the sum in x_n to a sum of integrals. Then show that the sequence (x_n) is monotone.

3. (a) Show there exists $\epsilon > 0$ such that $\log(1 - x) \geq -2x$ for $x \in [0, \epsilon)$.

   (b) Define $p_n = \prod_{i=1}^{n} \frac{2^i - 1}{2^i}$ for $n = 1, 2, \ldots$. Use the result in (a) to show that the limit $\lim_{n\to\infty} p_n$ exists and is positive.

4. A sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \ldots$ is said to be *uniformly equicontinuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon$ for all n whenever $|x - y| < \delta$.

   Suppose the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, \ldots$ is uniformly equicontinuous, and suppose the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$. Show that the function defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous.

5. Suppose $U \subset \mathbb{R}$ is open. Show that

$$U \subseteq \overline{U \cap \mathbb{Q}}.$$

   Here $\overline{V}$ denotes the closure of a set $V \subseteq R$, and $\mathbb{Q}$ denotes the set of rational numbers.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Probability

Wednesday, January 18, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. In a box there are 99 fair coins and 1 double-headed coin (i.e., it always comes up heads). Suppose that I randomly draw a coin, flip it n times, and it comes up heads every time. What is the smallest n such that, given this information, the probability that the coin is double-headed is at least $\frac{9}{10}$?

2. In a group of n people, suppose that each person has equal probability $\frac{1}{365}$ of having their birthday on any day of the year (assume Feb 29th is not possible), and the birthdays are independent. We say a birthday is *unique* if exactly one person in the group has that birthday. What is the expected number of unique birthdays, as a function of n?

3. 6 ducks and 6 geese randomly sit down in a circle, with each permutation equally likely. Compute the probability that there are at least 4 ducks in a row somewhere in the circle.

4. Let X_1, X_2, and X_3 be independent random variables drawn from the exponential distribution with mean 1. Compute the probability density function of X_2 conditional on the event $X_1 \leq X_2 \leq X_3$.

   (Equivalently, let $Y_1 \leq Y_2 \leq Y_3$ be X_1, X_2, X_3 sorted in increasing order. Compute the probability density function of Y_2.)

5. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ whose entries are independently drawn from a standard normal distribution $X_k \sim N(0,1)$. For $x \in \mathbb{R}^n$, let $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. For any $\epsilon > 0$, show that there exists a constant C_ϵ such that for all large enough n,

$$P(\|X\| - \sqrt{n} > C_\epsilon) < \epsilon.$$

Applied Mathematics and Statistics
Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Linear Algebra

Thursday, January 19, 2023

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $P_1, P_2 \in \mathbb{R}^{n\times n}$ be projection matrices. Show that if $P_1 - P_2$ is a projection matrix, then $\text{range}(P_2) \subseteq \text{range}(P_1)$.

2. Find the rank of
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

3. Consider the matrix A below:
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}.$$
If A is diagonalizable, find a diagonalizing matrix S such that $S^{-1}AS$ is diagonal. Otherwise, show that A is not diagonalizable.

4. Is there an invertible matrix $A \in \mathbb{R}^{n\times n}$ for any value of $n \geq 2$ for which every $(n-1)\times(n-1)$ minor (not necessarily principal) is equal to 1? Prove that no such matrix can exist, or find a matrix with this property.

5. Consider the real vector space V of polynomials with real coefficients having degree at most d, where $d \geq 2$ is an integer. Show that the subspaces S_1, S_2 satisfy $\dim(S_1 \cap S_2) \geq (d-1)/2$, where these are defined as:
   - $S_1 = \{p \in V : p(1) = 0\}$.
   - $S_2 = \{p \in V : p \text{ is even, i.e., } p(x) = p(-x) \text{ for all } x\}$.