Department of Applied Mathematics and Statistics
The Johns Hopkins University

# Introductory Examination–Summer Session
## Morning Exam–Real Analysis

Tuesday, August 23, 2022

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose we have a sequence of real numbers $a_k$ such that $\sum_{k=1}^{\infty} |a_k|$ converges. Prove that $\sum_{k=1}^{\infty} a_k^2$ converges. Also show that the converse is false.

*Solution:* Since $\sum_{k=1}^{\infty} |a_k|$ converges, $\lim_{k\to\infty} |a_k| = 0$. Therefore, there exists an $N$ such $|a_k| < 1$ for all $k \geq N$. Consequently, $0 \leq a_k^2 \leq |a_k| < 1$ for $k \geq N$ and

$$\begin{aligned} \sum_{k=1}^{\infty} a_k^2 &= \sum_{k=1}^{N-1} a_k^2 + \sum_{k=N}^{\infty} a_k^2 \\ &\leq \sum_{k=1}^{N-1} a_k^2 + \sum_{k=N}^{\infty} |a_k|, \end{aligned}$$

so $\sum_{k=1}^{\infty} a_k^2$ converges by comparison.

For showing the converse is false consider $a_k = 1/k$ for which $\sum_{k=1}^{\infty} |a_k| = +\infty$ but $\sum_{k=1}^{\infty} a_k^2 < +\infty$.

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2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Prove $e^{\int_0^1 f(x)\,dx} \leq \int_0^1 e^{f(x)}\,dx$.

Hint: First consider Riemann sum approximations.

*Solution:* We can compute $\int_0^1 f(x)\,dx$ as $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(\frac{j-1}{n})$ , and $\int_0^1 e^{f(x)}\,dx$ in similar fashion. Now,

$$\begin{aligned} e^{\sum_{j=1}^{n} \frac{1}{n} f(\frac{j-1}{n})} &= \left(e^{f(0)} e^{f(\frac{1}{n})} e^{f(\frac{2}{n})} \cdots e^{f(\frac{n-1}{n})}\right)^{\frac{1}{n}} \\ &\leq \frac{1}{n}\left(e^{f(0)} + e^{f(\frac{1}{n})} + e^{f(\frac{2}{n})} + \cdots + e^{f(\frac{n-1}{n})}\right) \end{aligned}$$

by the arithmetic–geometric means inequality. Since this holds for all $n$, passing to the limit gives us the result.

Alternative solution: This is an immediate consequence of Jensen's inequality applied to the exponential function and the random variable $f(U)$, where $U$ is distributed uniform$(0,1)$.

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3. Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ denote the standard normal probability density function, and let $g(x) = \frac{1}{2}e^{-|x|}$ denote the double exponential density function. Find the smallest value of $c > 0$ such that $f(x) \leq cg(x)$ for all $x \in \mathbb{R}$.

*Solution:* The condition

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \leq c\frac{1}{2}e^{-|x|} \text{ for all } x \in \mathbb{R}$$

can be rewritten as

$$\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2+|x|} \leq c \text{ for all } x \in \mathbb{R}$$

so the minimum value of $c$ is given by

$$\sup_x \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2+|x|}.$$

To find the supremum, we maximize the exponent $-\frac{1}{2}x^2+|x|$. By symmetry, it suffices to maximize over $x \geq 0$, and for such $x$

$$-\frac{1}{2}x^2 + |x| = -\frac{1}{2}x^2 + x = -\frac{1}{2}(x-1)^2 + \frac{1}{2}.$$

The maximum of this expression occurs when $x = 1$ yielding a maximum of $\frac{1}{2}$, giving

$$c = \frac{2}{\sqrt{2\pi}} e^{\frac{1}{2}} = \sqrt{\frac{2e}{\pi}}.$$

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4. Suppose $a_n > 0$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < +\infty$. Prove that $\sum_{n=1}^{\infty} a_n^{n/(n+1)} < +\infty$. Hint. Define

$$I = \left\{ n \ : \ a_n^{n/(n+1)} \leq 2a_n \right\},$$

and show that for $n \notin I$ we have $a_n^{n/(n+1)} < 1/2^n$.

*Solution:* If $n \notin I$ we have

$$a_n^{n/(n+1)} > 2a_n,$$

so that

$$a_n^n > 2^{n+1}a_n^{n+1} = 2^{n+1}a_n a_n^n,$$

and dividing both sides by $2^{n+1}a_n^n$ gives

$$a_n < 1/2^{n+1}.$$

Consequently

$$a_n^{n/(n+1)} < 1/2^n.$$

Breaking the sum of interest into two pieces,

$$\sum_{n=1}^{\infty} a_n^{n/(n+1)} = \sum_{n \in I} a_n^{n/(n+1)} + \sum_{n \notin I} a_n^{n/(n+1)}$$

$$\leq \sum_{n \in I} 2a_n + \sum_{n \notin I} 1/2^n$$

and these last two sums are finite.

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5. Suppose $g_n : [0,1] \to \mathbb{R}$ is continuous function for $n = 1, 2 \ldots$, with $g_n \to g$ uniformly, and define $f_n(x) = \int_{t=0}^{x} g_n(t)dt$. Show $f_n$ converges uniformly to a differentiable limit function $f$ and describe this function.

*Solution:* Since the functions $g_n$ are continuous, by uniform convergence the limit $g$ is also continuous and we can define define a function $f(x) = \int_{t=0}^{x} g(t)dt$. Observe that this function is differentiable with $f' = g$.

We proceed to show uniform convergence of $f_n$ to $f$, i.e.,

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0.$$

For this,

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} \left| \int_{t=0}^{x} g_n(t)dt - \int_{t=0}^{x} g(t)dt \right| = \sup_{x \in [0,1]} \left| \int_{t=0}^{x} g_n(t) - g(t)dt \right|$$

$$\leq \sup_{x \in [0,1]} \int_{t=0}^{x} |g_n(t) - g(t)|\, dt = \int_{t=0}^{1} |g_n(t) - g(t)|\, dt.$$

The last equality follows from nonnegativity of the integrand. By uniform convergence if $\epsilon > 0$ there exists $N$ such that for $n \geq N$ we have $\sup_{t \in [0,1]} |g_n(t) - g(t)| \leq \epsilon$ so for $n \geq N$ we have $\int_{t=0}^{1} |g_n(t) - g(t)| dt \leq \epsilon$.

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Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Afternoon Exam–Probability

Tuesday, August 23, 2022

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Five (5) balls are dropped on a table that has 7 holes. Assuming each ball falls independently into any of the 7 holes with equal probability, find the probability that there is at least one hole into which more than one ball falls.

*Solution:* Call the desired event $A$. Then $A^c$ is the event that each ball goes in a distinct hole, and

$$P(A^c) = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5}.$$

Therefore, $P(A) = 1 - \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \dfrac{2041}{2401} \approx 0.85.$

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2. $X$ and $Y$ are independent standard normals, i.e., each have density $\varphi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. Find the density of $U = \frac{X}{Y}$.

*Solution:* The joint pdf of $X$ and $Y$ is $f(x,y) = \varphi(x)\varphi(y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$ for $(x,y) \in \mathbb{R}^2$. We're told $U = \dfrac{X}{Y}$, so consider the mapping $(x,y) \mapsto (u,v)$ defined by $u = \dfrac{x}{y}$, $v = y$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. This mapping is invertible and

$$x = uv \quad \text{and} \quad y = v$$

is the inverse mapping. The Jacobian determinant of this inverse is

$$J := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} = v.$$

Therefore, the joint pdf of $U$ and $V$ is

$$g(u,v) = f(x,y) \cdot |J| = f(uv, v) \cdot |v| = \frac{|v|}{2\pi} e^{-\frac{1}{2}(u^2v^2+v^2)} = \frac{|v|}{2\pi} e^{-\frac{v^2[u^2+1]}{2}}.$$

We now find the marginal $g_U(u)$ for $U$: Fix any $u \in \mathbb{R}$; then

$$\begin{aligned} g_U(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |v| e^{-\frac{v^2[u^2+1]}{2}}\, dv \\ &= \frac{2}{2\pi}\int_0^{\infty} v\, e^{-\frac{v^2[u^2+1]}{2}}\, dv \quad \left(\text{next substitute } w = \tfrac{1+u^2}{2}v^2,\ dw = [1+u^2]v\, dv\right) \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+u^2}\, e^{-w}\, dw = \frac{1}{\pi(1+u^2)}, \end{aligned}$$

the *standard Cauchy density.*

3. Two (2) people each toss a fair coin until they observe the trial number of their first head. Let $X$ (resp., $Y$) represent the trial of person 1's (resp., person 2's) first head. Compute $E(X|X < Y)$, i.e., the conditional expectation of $X$ given $X < Y$.

*Solution:* First of all,

$$\begin{aligned} E(X|X<Y) &= \sum_{k=1}^{\infty} k \cdot P(X=k|X<Y) \\ &= \sum_{k=1}^{\infty} k \cdot \frac{P(X=k, X<Y)}{P(X<Y)} \\ &= \sum_{k=1}^{\infty} k \cdot \frac{P(X=k, k<Y)}{P(X<Y)} \\ &= \sum_{k=1}^{\infty} k \cdot \frac{P(X=k)P(Y>k)}{P(X<Y)}. \end{aligned}$$

Since $P(X < Y) = P(Y < X)$ we have $2P(X < Y) + P(X = Y) = 1$. Now,

$$P(X=Y) = \sum_{k=1}^{\infty} P(X=k, Y=k) = \sum_{k=1}^{\infty} P(X=k)P(Y=k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} = \frac{1}{3}.$$

Therefore, $P(X < Y) = (1 - \frac{1}{3})/2 = \frac{1}{3}$. Lastly, $P(X = k) = P(Y > k) = \left(\frac{1}{2}\right)^k$. Substituting this information into our calculation above we have

$$E(X|X<Y) = \sum_{k=1}^{\infty} k \cdot \frac{3}{4}\left(\frac{1}{4}\right)^{k-1} = \frac{4}{3},$$

since the summation represents the mean of a geometric random variable with success probability $\frac{3}{4}$.

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4. $X$ and $Y$ are independent random variables each having mean 0 and variance 1. Find a value $c$ such that $P[(X + Y)^2 \geq c]$ is at most .2.

*Solution:* Note that, by independence, $E[(X + Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[X]E[Y] + E[Y^2] = 2$. Therefore, by Markov's inequality,

$$P[(X+Y)^2 \geq c] \leq \frac{E[(X+Y)^2]}{c} = \frac{2}{c} \leq 0.2$$

exactly when $c \geq 10$.

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5. Let $A$ and $B$ be mutually exclusive events such that $P(A) = p$ and $P(B) = q$ with $0 < p + q < 1$. An experiment consists of repeated independent trials where on each trial we observe whether $A$, $B$, or $(A \cup B)^c$ occurred. Compute the probability that $A$ occurs before $B$.

*Solution:* If we let $A_i$, $B_i$ and $C_i$ represent the events of $A$, $B$ and $A^c \cap B^c$, respectively, occurring on trial $i$, then the event that $A$ occurs before $B$ is the event

$$A_1 \cup (C_1 \cap A_2) \cup (C_1 \cap C_2 \cap A_3) \cup \cdots = \bigcup_{i=1}^{\infty} \left( A_i \cap \bigcap_{k=1}^{i-1} C_k \right).$$

Since the events in this union are mutually exclusive and events with differing subscripts are independent, the probability of this event is

$$\begin{aligned} &= P(A_1) + P(C_1 \cap A_2) + P(C_1 \cap C_2 \cap A_3) + \cdots \\ &= p + (1 - p - q)p + (1 - p - q)^2 p + \cdots \\ &= \frac{p}{1 - (1 - p - q)} = \frac{p}{p + q}. \end{aligned}$$

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Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Linear Algebra

Wednesday, August 24, 2022

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

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5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Consider the symmetric tridiagonal matrix $A_n$ with 2 on the main diagonal and 1 on its first off-diagonal. For example, when $n = 4$, the matrix in question is

$$A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) Compute the determinant of $A_4$ above, and show that this matrix is invertible.

(b) Find a recurrence relation for the determinant of $A_n$, and solve it to find the general expression. Conclude that $A_n$ is invertible for all $n \geq 1$.

*Solution:* (a) Expanding by minors, we find that $\det(A_4) = 5$.

(b) An approach by induction works well, since

$$A_n = \begin{bmatrix} 2 & 1 & & \\ 1 & \ddots & & \\ & & A_{n-1} & \\ & & & \ddots \end{bmatrix}.$$

Now $\det(A_1) = 2$, and $\det(A_2) = 3$, so expanding by minors along the first column, we see that $\det(A_3) = 2\det(A_2) - 1\det(\tilde{A}_2)$, where $\tilde{A}_2$ is the matrix $A_2$ with its first row replaced by $[1, 0]$. This equals $2(3) - 1(2) = 4$. In general, $\det(A_n) = 2\det(A_{n-1}) - 1\det(\tilde{A}_{n-1})$, and $\det(\tilde{A}_n) = \det(A_{n-1})$. So we obtain $\det(A_n) = 2\det(A_{n-1}) - 1\det(A_{n-2})$, which is a second-order recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, with $a_1 = 2, a_2 = 3$. This has the simple solution $a_n = n + 1$, which one could easily guess from the first few values: indeed, we get $2n - (n - 1) = n + 1$, as claimed. So $\det(A_n) = n + 1$ for all $n$, and this determinant is never 0.

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2. Consider the following statement: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and suppose there are two distinct solutions $x_1, x_2 \in \mathbb{R}^n$ to the system $Ax = b$. Then there is some $c \in \mathbb{R}^m$ for which there is no solution to $Ax = c$.
Either prove that the statement is true for all $m$ and $n$, or find a counterexample for some $m$ and $n$.

*Solution:* Here is a simple example for the case $n = 2m$: Consider $A = [I_m, I_m]$, and any $b \in \mathbb{R}^m$. Then $[b^T, 0_m^T]^T$ and $[0_m^T, b^T]^T$ are distinct solutions to $Ax = b$, but $A$

clearly has full column rank. In particular, given any $c \in \mathbb{R}^m$, the vector $x = [c^T, 0_m^T]^T$ solves $Ax = c$.

For completeness, we now consider the case of general $m$ and $n$. If $n \le m$, then $\text{rank}(A) = n - \text{nullity}(A) < n \le m$, so there is some $c \in \mathbb{R}^m$ for which there is no solution to $Ax = c$. If $n > m$, consider $A = [I \,|\, A']$, where $A' \in \mathbb{R}^{m \times (n-m)}$ is any matrix. Then $\text{nullity}(A) = n - \text{rank}(A) = n - m > 0$, so the nullspace of $A$ is nontrivial; further, $\text{im}(A) = \mathbb{R}^m$. Thus for *any* $b \in \mathbb{R}^m$ there are two distinct solutions to the system $Ax = b$. But there is no $c \in \mathbb{R}^m$ for which there is no solution to $Ax = c$, so we have produced the desired counterexample.

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3. Suppose $A \in \mathbb{R}^{n \times n}$ satisfies $A^2 = 2A - I$. Show that the characteristic polynomial of $A$ is $(t-1)^n$.

*Solution:* The given condition says that $(A - I)^2 = 0$. Thus, the minimal polynomial of $A$ divides $(t-1)^2$, so the only possible eigenvalue of $A$ is 1. Since $A$ has order $n$, its characteristic polynomial must be $(t-1)^n$.

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4. Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices, and suppose $B$ is positive semidefinite. We say that $A$ is *positive semidefinite with respect to* $B$ if for any $x \in \mathbb{R}^n$ with $x^T B x \neq 0$, we have $x^T A x \ge 0$. Let $B^{1/2}$ be the unique symmetric positive semidefinite square root of $B$.

   Show that if $A$ and $B$ are symmetric, $B$ is positive semidefinite, and $A$ is positive semidefinite with respect to $B$, then $B^{1/2} A B^{1/2}$ is positive semidefinite. [Hint: Reduce to the case where $B$ is diagonal, then consider the quadratic form associated to $B^{1/2} A B^{1/2}$.]

*Solution:* Let $A$ be symmetric and let $B$ be symmetric positive definite with spectral decomposition $B = U \Lambda U^T$, where $U$ is a real orthogonal matrix and $\Lambda$ is a diagonal matrix with nonnegative entries. It is straightforward to argue that $A$ is positive semidefinite with respect to $B$ if and only if the symmetric matrix $UAU^T$ is positive semidefinite with respect to $\Lambda$. Another straightforward argument then establishes that it is sufficient to prove the desired result in the case that $B$ is a diagonal positive semidefinite matrix. For this, we may suppose that the first $k$ diagonal entries of $B$ are strictly positive and the last $n-k$ diagonal entries vanish, where $0 \le k \le n$.

To prove that $B^{1/2} A B^{1/2}$ is positive semidefinite, we show that $y^T B^{1/2} A B^{1/2} y \ge 0$ for any vector $y$. This nonnegativity is an immediate consequence of the positive semidefiniteness of $A$ with respect to $B$. Indeed, if $x = B^{1/2}y$, then $x^T Bx = y^T B^2 y \neq 0$ if and only if at least one of $y_1, \dots, y_k$ is nonzero; on the other hand, if $y_1 = \dots = y_k = 0$, then $B^{1/2}y = 0$ and $y^T B^{1/2} AB^{1/2} y = 0$.

Remark: The converse holds under the condition that $\mathrm{im}(A) \subseteq \mathrm{im}(B)$. If $B^{1/2}AB^{1/2}$ is positive semidefinite, then $y^T Ay \geq 0$ for any $y \in \mathrm{im}(B)$. From the condition that $\mathrm{im}(A) \subseteq \mathrm{im}(B)$, we see that $\mathrm{null}(B) = \mathrm{im}(B)^\perp \subseteq \mathrm{im}(A)^\perp = \mathrm{null}(A)$. Then for any vector $x \in \mathbb{R}^n$ with $x^T Bx \neq 0$, decomposing $x = y + z$ with $y \in \mathrm{im}(B)$ and $z \in \mathrm{null}(B)$, we must have $z \in \mathrm{null}(A)$, too. So $x^T Ax = (y+z)^T A(y+z) = y^T Ay \geq 0$ since $y \in \mathrm{im}(B)$.

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5. Consider a matrix of the form

$$A(x) = \begin{bmatrix} 1 & x^T \\ x & I_n \end{bmatrix},$$

where $x \in \mathbb{R}^n$. Give a necessary and sufficient condition for $A(x)$ to be invertible.

*Solution:* The necessary and sufficient condition for $A(x)$ to be invertible is that $\|x\| \neq 1$.

*Method 1:* Suppose $\|x\| = 1$. Then $y = [1, -x^T]^T$ satisfies $A(x)y = 0$, so $A(x)$ is not invertible. If $\|x\| \neq 1$, we note that $A(0) = I_{n+1}$ is clearly invertible, so we may consider $\|x\| \neq 1$, $x \neq 0$. For any $y = [c, v^T]^T$, if $A(x)y = 0$, then (i) $c + x^T v = 0$, which implies that $c = -x^T v$; and (ii) $cx + v = 0$. So $(x^T v)x = v$, implying that $v$ and $x$ are linearly dependent. If $v = ax$, this last equation reads $a\|x\|^2 x = ax$, or equivalently, $a(\|x\|^2 - 1)x = 0$, which is impossible unless $a = 0$ since $\|x\| \neq 1$ and $x \neq 0$. But $a = 0$ implies that $v = 0$, so $c = 0$ and $y = 0$. That is, $A(x)$ is nonsingular. So $A(x)$ is nonsingular if and only if $\|x\| \neq 1$.

*Method 2:* Clearly, the last $n$ columns of $A(x)$ are linearly independent, so $A(x)$ is not invertible if and only if the first column can be expressed as a linear combination of the others. Letting $c_0, \dots, c_n$ denote the columns of $A(x)$ then solving

$$c_0 = u_1 c_1 + \dots + u_n c_n$$

for $u_i \in \mathbb{R}$, we see that a solution exists if and only if $u_i = x_i$ for $i = 1, \dots, n$, and $1 = \sum_{i=1}^n u_i x_i$, i.e., if and only if $1 = \sum_{i=1}^n x_i^2$. So $A(x)$ is nonsingular if and only if $\|x\| \neq 1$.

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