Department of Applied Mathematics and Statistics The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION MORNING EXAM–REAL ANALYSIS

Tuesday, August 23, 2022

Instructions: Read carefully!

- 1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
- 2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear**, **logically justified steps**. The grading will reflect that broader purpose.
- 3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
- 4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
- 5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
- 6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
- 7. No calculators of any sort are needed or permitted.

- 1. Suppose we have a sequence of real numbers a_k such that $\sum_{k=1}^{\infty} |a_k|$ converges. Prove that $\sum_{k=1}^{\infty} a_k^2$ converges. Also show that the converse is false.
- 2. Suppose $f: [0,1] \to \mathbb{R}$ is continuous. Prove $e^{\int_0^1 f(x) dx} \leq \int_0^1 e^{f(x)} dx$. Hint: First consider Riemann sum approximations.
- 3. Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ denote the standard normal probability density function, and let $g(x) = \frac{1}{2}e^{-|x|}$ denote the double exponential density function. Find the smallest value of c > 0 such that $f(x) \le cg(x)$ for all $x \in \mathbb{R}$.
- 4. Suppose $a_n > 0$ for n = 1, 2, ... and $\sum_{n=1}^{\infty} a_n < +\infty$. Prove that $\sum_{n=1}^{\infty} a_n^{n/(n+1)} < +\infty$. Hint. Define

$$I = \left\{ n : a_n^{n/(n+1)} \le 2a_n \right\},\,$$

and show that for $n \notin I$ we have $a_n^{n/(n+1)} < 1/2^n$.

5. Suppose $g_n : [0,1] \to \mathbb{R}$ is continuous function for n = 1, 2..., with $g_n \to g$ uniformly, and define $f_n(x) = \int_{t=0}^x g_n(t) dt$. Show f_n converges uniformly to a differentiable limit function f and describe this function.

Department of Applied Mathematics and Statistics The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION AFTERNOON EXAM–PROBABILITY

Tuesday, August 23, 2022

Instructions: Read carefully!

- 1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
- 2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear**, **logically justified steps**. The grading will reflect that broader purpose.
- 3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
- 4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
- 5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
- 6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
- 7. No calculators of any sort are needed or permitted.

- 1. Five (5) balls are dropped on a table that has 7 holes. Assuming each ball falls independently into any of the 7 holes with equal probability, find the probability that there is at least one hole into which more than one ball falls.
- 2. X and Y are independent standard normals, i.e., each have density $\varphi(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. Find the density of $U = \frac{X}{Y}$.
- 3. Two (2) people each toss a fair coin until they observe the trial number of their first head. Let X (resp., Y) represent the trial of person 1's (resp., person 2's) first head. Compute E(X|X < Y), i.e., the conditional expectation of X given X < Y.
- 4. X and Y are independent random variables each having mean 0 and variance 1. Find a value c such that $P[(X + Y)^2 \ge c]$ is at most .2.
- 5. Let A and B be mutually exclusive events such that P(A) = p and P(B) = q with $0 . An experiment consists of repeated independent trials where on each trial we observe whether A, B, or <math>(A \cup B)^c$ occurred. Compute the probability that A occurs before B.

Department of Applied Mathematics and Statistics The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION MORNING EXAM–LINEAR ALGEBRA

Wednesday, August 24, 2022

Instructions: Read carefully!

- 1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
- 2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear**, **logically justified steps**. The grading will reflect that broader purpose.
- 3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
- 4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
- 5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
- 6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
- 7. No calculators of any sort are needed or permitted.

1. Consider the symmetric tridiagonal matrix A_n with 2 on the main diagonal and 1 on its first off-diagonal. For example, when n = 4, the matrix in question is

$$A_4 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

- (a) Compute the determinant of A_4 above, and show that this matrix is invertible.
- (b) Find a recurrence relation for the determinant of A_n , and solve it to find the general expression. Conclude that A_n is invertible for all $n \ge 1$.
- 2. Consider the following statement: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and suppose there are two distinct solutions $x_1, x_2 \in \mathbb{R}^n$ to the system Ax = b. Then there is some $c \in \mathbb{R}^m$ for which there is no solution to Ax = c. Either prove that the statement is true for all m and n, or find a counterexample for some m and n.
- 3. Suppose $A \in \mathbb{R}^{n \times n}$ satisfies $A^2 = 2A I$. Show that the characteristic polynomial of A is $(t-1)^n$.
- 4. Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices, and suppose B is positive semidefinite. We say that A is *positive semidefinite with respect to* B if for any $x \in \mathbb{R}^n$ with $x^T B x \neq 0$, we have $x^T A x \geq 0$. Let $B^{1/2}$ be the unique symmetric positive semidefinite square root of B.

Show that if A and B are symmetric, B is positive semidefinite, and A is positive semidefinite with respect to B, then $B^{1/2}AB^{1/2}$ is positive semidefinite. [Hint: Reduce to the case where B is diagonal, then consider the quadratic form associated to $B^{1/2}AB^{1/2}$.]

5. Consider a matrix of the form

$$A(x) = \begin{bmatrix} 1 & x^T \\ x & I_n \end{bmatrix},$$

where $x \in \mathbb{R}^n$. Give a necessary and sufficient condition for A(x) to be invertible.