Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–REAL ANALYSIS

Tuesday, January 18, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function. Show that there exists x such that $f(x^2) < f(x)^2 + \frac{1}{4}$.

Solution: If, on the contrary, we have $f(x^2) \geq f(x)^2 + \frac{1}{4}$ for all x, then in particular we have $f(0) \geq f(0)^2 + \frac{1}{4}$ and $f(1) \geq f(1)^2 + \frac{1}{4}$. But then

$$0 \geq f(0)^2 - f(0) + \frac{1}{4} = \left(f(0) - \frac{1}{2}\right)^2,$$

so $f(0) = \frac{1}{2}$. Similarly, if $f(1) \geq f(1)^2 + \frac{1}{4}$, then $f(1) = \frac{1}{2}$. Since f is a one-to-one function we have a contradiction.

---

2. Show that

$$\frac{1}{n}\sum_{k=1}^{n} \log k \leq \log(n+1) - \log 2$$

for any positive integer n.

Solution: We have

$$\frac{1}{n}\sum_{k=1}^{n} \log k = \frac{1}{n}\log\left(\prod_{k=1}^{n} k\right) = \log\left(\prod_{k=1}^{n} k^{1/n}\right),$$

and by the arithmetic/geometric-mean inequality and monotonicity of log this is bounded above by

$$\log\left(\frac{1}{n}\sum_{k=1}^{n} k\right) = \log\left(\frac{n+1}{2}\right).$$

---

3. For a real-valued sequence $x_1, x_2, \ldots$ define the partial sums

$$S_n = \sum_{i=1}^{n} x_i \text{ for } n = 0, 1, 2, \ldots$$

(with $S_0 := 0$). Assume $\lim_{n\to\infty} S_n = s$ where $s \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} kx_k = 0.$$

Hint: Start by proving the following summation-by-parts identity:

$$\frac{1}{n}\sum_{k=1}^{n} kx_k = S_n - \frac{1}{n}\sum_{k=0}^{n-1} S_k.$$

Solution: We start by proving the identity in the hint:

$$\sum_{k=1}^{n} kx_k = \sum_{k=1}^{n}\left(\sum_{i=0}^{k-1} 1\right) x_k = \sum_{i=0}^{n-1}\left(\sum_{k=i+1}^{n} x_k\right) = \sum_{i=0}^{n-1}(S_n - S_i) = nS_n - \sum_{k=0}^{n-1} S_k.$$

Now apply Cesàro's theorem to the identity to conclude that

$$\frac{1}{n}\sum_{k=1}^{n} kx_k \to s - s = 0,$$

as desired.

---

4. Suppose f and g are continuous real-valued functions defined on $[0,1]$ with

$$\max_{x\in[0,1]} f(x) = \max_{x\in[0,1]} g(x).$$

Show there exists $x^* \in [0,1]$ such that

$$f(x^*) = g(x^*).$$

Solution: Let M be the common maximum value of the two functions. Since f and g are continuous, there exist $u, v \in [0,1]$ such that $f(u) = M$ and $g(v) = M$. Choose one such pair (u, v). If $u = v$ we can take x^* to be this common point. If $f(v) = M$ we can take $x^* = v$ and if $g(u) = M$ we can take $x^* = u$. It remains to consider the case when $u \neq v$, $f(v) < M$, and $g(u) < M$. Then we have $f(u) - g(u) = M - g(u) > 0$ and $f(v) - g(v) = f(v) - M < 0$. Since $f - g$ is continuous, by the intermediate value theorem there exists x^* between u and v such that $f(x^*) - g(x^*) = 0$.

---

5. Let s_n denote the n^{th} partial sum of an alternating series, i.e., $s_n = \sum_{i=1}^{n}(-1)^{i+1}u_i$, where $u_n, n = 1, 2, \ldots,$ is a nonnegative monotone non-increasing sequence satisfying $\lim_{n\to\infty} u_n = 0$. Prove that the sequence $s_n, n = 1, 2, \ldots,$ converges.

Solution: We are told $s_N = \sum_{n=1}^{N}(-1)^{n+1}u_n$, where $u_n \geq 0$ for all n with u_n monotone non-increasing and $\lim_{n\to\infty} u_n = 0$.

Observe that $s_{2n} - s_{2n-2} = u_{2n-1} - u_{2n} \geq 0$ for all n. This shows that s_{2n} forms a monotone non-decreasing sequence. In addition,

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2n-1} - u_{2n})$$

so $s_{2n} \leq u_1$ for all n.

We have a bounded monotone sequence and therefore it converges, say

$$\lim_{n\to\infty} s_{2n} = s.$$

On the other hand, $s_{2n+1} = s_{2n} + u_{2n+1}$. Since $\lim_{n\to\infty} u_n = 0$, both sequences on the right converge and we conclude that

$$\lim_{n\to\infty} s_{2n+1} = s + 0 = s.$$

Therefore

$$\lim_{n\to\infty} s_n = s.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
AFTERNOON EXAM–PROBABILITY

Tuesday, January 18, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. In how many ways can 10 identical tokens be distributed among 4 children if the eldest must receive at least 2 of these tokens?

Solution: If we think of the distribution of these 10 tokens among the 4 children as a vector of nonnegative integers of length 4—so that (x_1, x_2, x_3, x_4) means the youngest receives x_1 tokens, the second youngest receives x_2 tokens, and so on—then we want to count the number of such vectors whose last entry is at least 2 and whose entries sum to 10. This is the same as giving the eldest 2 tokens, and counting the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 8$. This is just the number of weak compositions of the integer 8, which is $\binom{8+4-1}{4-1} = \binom{11}{3} = 165$.

---

2. There are 6 people (numbered 1, 2, ..., 6) in a room. Among these 6 people, 3 are left-handed and the others are right-handed. Independently for each i from 1 through 6, you toss a fair coin. If the coin comes up heads you shake person i's hand; otherwise, you do not shake their hand. Given you shook the hands of all the left-handed people, compute the probability you tossed exactly k heads for each integer k from 0 through 6.

Solution: Without loss of generality let's assume persons 1,2, and 3 are left-handed. We can recast the problem as follows: Given that the first three tosses of a fair coin are heads, what's the probability we toss k heads total in six tosses? If $k = 0, 1, 2$ the answer is 0. If $k = 3, 4, 5$, or 6, then this is the same as tossing $k - 3$ heads in 3 tosses, the probability of which is $\frac{\binom{3}{k-3}}{8}$. Explicitly, the answers are 0, 0, 0, 1/8, 3/8, 3/8, 1/8 for $k = 0, 1, 2, 3, 4, 5, 6$, respectively.

---

3. Suppose X is uniformly distributed over the unit interval $[0, 1]$. Derive the pdf of $U = \frac{X}{1+X}$.

Solution: The distribution of U is concentrated on the interval $[0, \frac{1}{2}]$. So, for $0 \leq u \leq \frac{1}{2}$, the cdf of U is

$$F_U(u) = P\Big(\frac{X}{1+X} \leq u\Big) = P(X \leq u + uX) = P\Big(X \leq \frac{u}{1-u}\Big) = \frac{u}{1-u}.$$

Consequently, the pdf of U is

$$f(u) = \begin{cases} \frac{1}{(1-u)^2} & \text{if } 0 \le u \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} .$$

---

4. We have a box filled with $2m$ marbles: two marbles in each of m different colors. Uniformly at random and without replacement, someone selects k marbles, where $1 \le k \le 2m$. Compute the expected value of the number of colors selected.

Solution: For $i = 1, 2, \ldots, m$, we let $X_i = 1$ if color i is in the selection and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{m} X_i$ counts the number of colors in the selection. Moreover, by linearity of expectation,

$$\begin{aligned} E(X) = \sum_{i=1}^{m} E(X_i) &= \sum_{i=1}^{m} P(X_i = 1) = \sum_{i=1}^{m} (1 - P(X_i = 0)) \\ &= \sum_{i=1}^{m} \left(1 - \frac{\binom{2}{0}\binom{2m-2}{k}}{\binom{2m}{k}}\right) = m\left(1 - \frac{\binom{2m-2}{k}}{\binom{2m}{k}}\right). \end{aligned}$$

---

5. The following function is known to be a moment generating function:

$$M(\theta) = (1 + \theta^2)e^{\theta^2/2}, \quad -\infty < \theta < \infty.$$

Suppose X is a random variable having $M(\theta)$ as its moment generating function. For each integer $k \ge 1$, compute the kth moment $E(X^k)$.

Solution: Expand the moment generating function:

$$
\begin{aligned}
M(\theta) = (1+\theta^2)e^{\theta^2/2} &= (1+\theta^2)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\theta^2}{2}\right)^m = (1+\theta^2)\sum_{m=0}^{\infty}\frac{1}{m!2^m}\theta^{2m} \\
&= \sum_{m=0}^{\infty}\frac{1}{m!2^m}\theta^{2m} + \sum_{m=0}^{\infty}\frac{1}{m!2^m}\theta^{2(m+1)} \\
&= 1 + \sum_{m=1}^{\infty}\frac{1}{m!2^m}\theta^{2m} + \sum_{m=1}^{\infty}\frac{1}{(m-1)!2^{m-1}}\theta^{2m} \\
&= 1 + \sum_{m=1}^{\infty}\frac{2m+1}{m!2^m}\theta^{2m} \\
&= 1 + \sum_{m=1}^{\infty}\frac{(2m+1)!}{m!2^m}\frac{\theta^{2m}}{(2m)!}.
\end{aligned}
$$

From here we can directly read off the moments. Since $M(\theta)$ is an even function, $E(X^k) = 0$ when $k \geq 1$ is odd. When $k = 2m$ is even, $E(X^{2m}) = \frac{(2m+1)!}{m!2^m}$.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–LINEAR ALGEBRA

Wednesday, January 19, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A positive semidefinite matrix $A \in \mathbb{R}^{n\times n}$ has a Cholesky factorization as $A = LL^T$, where $L \in \mathbb{R}^{n\times n}$ is lower triangular. Show that when A is positive definite, it also has a *loChesky* factorization as $A = UU^T$, where $U \in \mathbb{R}^{n\times n}$ is upper triangular.

   Hint: You may use without proof that the inverse of an upper triangular matrix is also upper triangular.

   Solution: A^{-1} is positive definite, so it has a Cholesky factorization as $A^{-1} = LL^T$. We see that L and L^T are also invertible, since $\det A^{-1} = (\det L)\left(\det L^T\right)$, and thus neither of $\det L$ and $\det L^T$ may equal 0. Then $A = (L^T)^{-1}L^{-1}$, where L^T is an upper triangular matrix, and thus $(L^T)^{-1}$ is, too. Since $(L^T)^{-1} = (L^{-1})^T$, this gives the desired factorization as $A = UU^T$ with $U = (L^{-1})^T$.

---

2. Suppose A and B are symmetric positive definite $n \times n$ real matrices.

   (a) Is it true that A and B must commute? If so, prove it or give a counterexample.

   (b) Assume A and B commute and $x^TAx \leq x^TBx$ for all $x \in \mathbb{R}^n$. Show that $\det(A) \leq \det(B)$. You may suppose that A has distinct eigenvalues if needed for your proof.

   Solution: (a) A and B need not commute and it is not very challenging to come up with examples for $n = 2$. For example, for $n = 2$ consider

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

   and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

   both of which are symmetric and strictly diagonally dominant, hence positive definite. But $(AB)_{12} = 2$ while $(BA)_{12} = 1$, so A and B do not commute. If $n = 1$, then A and B must of course commute. If $n \geq 3$, then a counterexample is obtained by taking the direct sum of each A and B with the $(n-2)$-dimensional identity matrix.

   (b) Approach 1 (not using distinct eigenvalues, and not using commutativity!): The quadratic-forms inequality shows that $B - A$ is symmetric positive semidefinite. The eigenvalues of A, $B - A$, and B are all nonnegative real numbers. Order the eigenvalues of A as $\lambda_1 \leq \cdots \leq \lambda_n$ and those of B as $\omega_1 \leq \cdots \leq \omega_n$. By Weyl's inequalities, for each $i = 1, \ldots, n$ we have $\omega_i \geq \lambda_i + \mu_1 \geq \lambda_i > 0$, where μ_1 is the smallest eigenvalue of $B - A$. Since $\det \Lambda = \prod_{i=1}^n \lambda_i$ and $\det B = \prod_{i=1}^n \omega_i$, the result follows.

Approach 2 (not using distinct eigenvalues): Since A and B are real and symmetric and they commute, they can be simultaneously orthogonally diagonalized, i.e., there exists a real orthogonal matrix P such that $P^T AP = \Lambda$ and $P^T BP = \Omega$, where Λ is a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$ and Ω is a diagonal matrix with diagonal entries $\omega_1, \ldots, \omega_n$. Taking $x = Pe^{(i)}$ where $e^{(i)}$ is the standard unit vector we obtain

$$x^T Ax = e^{(i)T} P^T P \Lambda P^T P e^{(i)} = e^{(i)T} \Lambda e^{(i)} = \lambda_i,$$

and similarly $x^T Bx = \omega_i$. So $\lambda_i \leq \omega_i$ for $i = 1, \ldots, n$. Also, since A and B are positive definite, we have $\lambda_i > 0$ and $\omega_i > 0$ for $i = 1, \ldots, n$. The proof now proceeds as in Approach 1.

Approach 3 (using distinct eigenvalues): A and B are commuting diagonalizable matrices, hence simultaneously diagonalizable, so there exists an invertible matrix S with $S^{-1}AS = \Lambda$ and $S^{-1}BS = \Omega$. The first equation may be re-written as $AS = S\Lambda$, so we see that the ith column of S is an eigenvector of A with eigenvalue λ_i. But since A has distinct eigenvalues, the eigenspace associated to each λ_i is one-dimensional. Further, since A has a set of n orthonormal eigenvectors $\{p_1, \ldots, p_n\}$, we see that each column of S must be a nonzero scalar multiple of one of these p_i's, and thus $S = PD$, where D is a diagonal matrix with nonzero diagonal entries and P is the matrix whose columns are $p_1, \ldots, p_n$. Then $A = S\Lambda S^{-1} = PD\Lambda D^{-1}P^{-1} = P\Lambda P^T$, where $P^{-1} = P^T$ since the columns of P are an orthonormal set, and similarly $B = P\Omega P^T$. We may now proceed as in Approach 2.

---

3. Suppose $A \in \mathbb{R}^{5\times 5}$ is an invertible matrix such that A and A^{-1} are similar.

   (a) Show that $A - A^{-1}$ is similar to $A^{-1} - A$.

   (b) Use this to conclude that at least one of ± 1 is an eigenvalue of A.

Solution: (a) Using the given condition that $A = SA^{-1}S^{-1}$ for some invertible $S \in \mathbb{R}^{n\times n}$, we may invert both sides of this equation to see that we also have $A^{-1} = SAS^{-1}$. Combining these, we have

$$A - A^{-1} = SA^{-1}S^{-1} - SAS^{-1} = S(A^{-1} - A)S^{-1}.$$

(b) We want to show that at least one of $A - I$ and $A + I$ is singular; or equivalently, that $(A - I)(A + I) = A^2 - I = A(A - A^{-1})$ is singular; or, equivalently, that $A - A^{-1}$ is singular. By part (a), each eigenvalue of $A - A^{-1}$ is also an eigenvalue of $A^{-1} - A = -(A - A^{-1})$, with matching multiplicities. That is, the multiplicity of any eigenvalue λ of $A - A^{-1}$ is the same as the multiplicity of $-\lambda$ as an eigenvalue of $A - A^{-1}$. Since the order of A is 5, which is odd, there must be some eigenvalue of $A - A^{-1}$ with $\lambda = -\lambda$, meaning that $\lambda = 0$.

4. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a symmetric real matrix, and suppose that the following matrices have the same eigenvalues:

$$A \odot A = \begin{bmatrix} a^2 & b^2 \\ b^2 & c^2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}^2.$$

Show that A is diagonal.

Solution: Since the eigenvalues are the same, the traces are equal. This gives $a^2+c^2 = a^2+2b^2+c^2$, when we carry out the multiplication in A^2. Thus $b^2 = 0$, implying that $b = 0$ and A is diagonal.

---

5. Compute the orthogonal projection matrix onto the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: We first find an orthonormal basis for this subspace and its orthogonal complement using Gram–Schmidt orthogonalization:

$$\begin{aligned}
w_1 &= [1,1,1]^T \\
v_1 &= [1,1,1]^T/\sqrt{3} \\
w_2 &= [2,1,1]^T - \langle [2,1,1]^T, v_1 \rangle v_1 = [2,1,1]^T - [4/3,4/3,4/3]^T = [2/3,-1/3,-1/3]^T \\
v_2 &= [2,-1,-1]^T/\sqrt{6} \\
v_3 &= [0,1,-1]^T/\sqrt{2}
\end{aligned}$$

Method 1:

$$P = v_1 v_1^T + v_2 v_2^T = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Method 2:

$$P = I - v_3 v_3^T = I - \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$
