Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Real Analysis

Tuesday, January 18, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a one-to-one function. Show that there exists x such that $f(x^2) < f(x)^2 + \frac{1}{4}$.

2. Show that

$$\frac{1}{n}\sum_{k=1}^{n} \log k \le \log(n+1) - \log 2$$

for any positive integer n.

3. For a real-valued sequence $x_1, x_2, \ldots$ define the partial sums

$$S_n = \sum_{i=1}^{n} x_i \text{ for } n = 0, 1, 2, \ldots$$

(with $S_0 := 0$). Assume $\lim_{n\to\infty} S_n = s$ where $s \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} kx_k = 0.$$

Hint: Start by proving the following summation-by-parts identity:

$$\frac{1}{n}\sum_{k=1}^{n} kx_k = S_n - \frac{1}{n}\sum_{k=0}^{n-1} S_k.$$

4. Suppose f and g are continuous real-valued functions defined on $[0, 1]$ with

$$\max_{x\in[0,1]} f(x) = \max_{x\in[0,1]} g(x).$$

Show there exists $x^* \in [0, 1]$ such that

$$f(x^*) = g(x^*).$$

5. Let s_n denote the n^{th} partial sum of an alternating series, i.e., $s_n = \sum_{i=1}^{n}(-1)^{i+1}u_i$, where $u_n, n = 1, 2, \ldots,$ is a nonnegative monotone non-increasing sequence satisfying $\lim_{n\to\infty} u_n = 0$. Prove that the sequence $s_n, n = 1, 2, \ldots,$ converges.

Department of Applied Mathematics and Statistics
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Introductory Examination–Winter Session
Afternoon Exam–Probability

Tuesday, January 18, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. In how many ways can 10 identical tokens be distributed among 4 children if the eldest must receive at least 2 of these tokens?

2. There are 6 people (numbered 1, 2, ..., 6) in a room. Among these 6 people, 3 are left-handed and the others are right-handed. Independently for each i from 1 through 6, you toss a fair coin. If the coin comes up heads you shake person i's hand; otherwise, you do not shake their hand. Given you shook the hands of all the left-handed people, compute the probability you tossed exactly k heads for each integer k from 0 through 6.

3. Suppose X is uniformly distributed over the unit interval $[0, 1]$. Derive the pdf of $U = \frac{X}{1+X}$.

4. We have a box filled with $2m$ marbles: two marbles in each of m different colors. Uniformly at random and without replacement, someone selects k marbles, where $1 \leq k \leq 2m$. Compute the expected value of the number of colors selected.

5. The following function is known to be a moment generating function:

$$M(\theta) = (1 + \theta^2)e^{\theta^2/2}, \quad -\infty < \theta < \infty.$$

Suppose X is a random variable having $M(\theta)$ as its moment generating function. For each integer $k \geq 1$, compute the kth moment $E(X^k)$.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–LINEAR ALGEBRA

Wednesday, January 19, 2022

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A positive semidefinite matrix $A \in \mathbb{R}^{n\times n}$ has a Cholesky factorization as $A = LL^T$, where $L \in \mathbb{R}^{n\times n}$ is lower triangular. Show that when A is positive definite, it also has a *loChesky* factorization as $A = UU^T$, where $U \in \mathbb{R}^{n\times n}$ is upper triangular.

   Hint: You may use without proof that the inverse of an upper triangular matrix is also upper triangular.

2. Suppose A and B are symmetric positive definite $n \times n$ real matrices.

   (a) Is it true that A and B must commute? If so, prove it or give a counterexample.

   (b) Assume A and B commute and $x^T Ax \leq x^T Bx$ for all $x \in \mathbb{R}^n$. Show that $\det(A) \leq \det(B)$. You may suppose that A has distinct eigenvalues if needed for your proof.

3. Suppose $A \in \mathbb{R}^{5\times 5}$ is an invertible matrix such that A and A^{-1} are similar.

   (a) Show that $A - A^{-1}$ is similar to $A^{-1} - A$.

   (b) Use this to conclude that at least one of ± 1 is an eigenvalue of A.

4. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a symmetric real matrix, and suppose that the following matrices have the same eigenvalues:

$$A \odot A = \begin{bmatrix} a^2 & b^2 \\ b^2 & c^2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}^2.$$

   Show that A is diagonal.

5. Compute the orthogonal projection matrix onto the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$