Department of Applied Mathematics and Statistics
The Johns Hopkins University

# Introductory Examination–Winter Session
## Morning Exam–Real Analysis

Tuesday, January 22, 2019

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $K$ a compact subset of the real line. Prove: $f(K) = \{f(x) \in \mathbb{R} : x \in K\}$ is compact.

*Solution:* Suppose we have any open cover of $f(K)$, say, $\cup \mathcal{O}_\alpha$, where $\alpha$ belongs to some (possibly uncountable) index set. Since $f$ is continuous, the inverse image of each $\mathcal{O}_\alpha$ is also open; therefore, $\cup f^{-1}[\mathcal{O}_\alpha]$ forms an open cover of $K$. Now, $K$ is compact so that there is a finite subcover, say $f^{-1}[\mathcal{O}_1], f^{-1}[\mathcal{O}_2], \ldots, f^{-1}[\mathcal{O}_n]$ which covers $K$. Consequently, $\mathcal{O}_1, \mathcal{O}_2 \ldots, \mathcal{O}_n$ is still a cover of $f(K)$ and $f(K)$ is compact.

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2. Prove that there exists $\varepsilon > 0$ such that

$$\cos(\sin x) > \frac{1}{2}\left(1 + \cos^2(x)\right)$$

whenever $0 < |x| < \varepsilon$.

*Solution:* Since $\cos(\sin(-x)) = \cos(\sin(x))$ it is enough to show that there exists $\varepsilon > 0$ such that $\cos(\sin(x)) > \frac{1}{2}(1 + \cos^2(x))$ whenever $0 < x < \varepsilon$. By Taylor's theorem, for $u > 0$, $\cos(u) = 1 - \frac{u^2}{2} + \cos(\xi)\frac{u^4}{4!}$ for some $0 < \xi < u$. Since $\cos(\xi) > 0$ for $0 < \xi < 1 < \frac{\pi}{2}$ it follows that $\cos(u) > 1 - \frac{u^2}{2}$ when $0 < u < 1$. Now, for $0 < x < 1 < \frac{\pi}{2}$, $0 < \sin(x) < 1$. Therefore, for $0 < x < 1$,

$$\cos(\sin(x)) > 1 - \frac{1}{2}\sin^2(x) = 1 - \frac{1}{2}(1 - \cos^2(x)) = \frac{1}{2}\left(1 + \cos^2(x)\right),$$

which was to be shown.

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3. Let $f, g : [0, \infty) \to \mathbb{R}$ be uniformly continuous functions. Must their product be uniformly continuous? If so, prove it. If not, give a counterexample.

*Solution:* No, the product of uniformly continuous functions defined on the non-negative reals need not be uniformly continuous. For example, one can choose $f(x) = g(x) = x$ and set out to show $f(x)g(x) = x^2$ is *not* uniformly continuous by noting that for any fixed $\delta > 0$, $|f(x) - f(x - \delta)| = |2\delta x - \delta^2|$ diverges to infinity as $x \to \infty$ and therefore can never remain small for all $x$.

Alternatively, we can consider the functions $f(x) = x$ and $g(x) = \sin(x)$ which are both clearly uniformly continuous. Fix $\varepsilon > 0$. To see their product $h(x) = x\sin(x)$ is not uniformly continuous we consider the behavior of $h$ near its roots: $x_n = \frac{(2n+1)\pi}{2}$. For any $\delta > 0$, $|h(x_n + \delta) - h(x_n)| = (\frac{(2n+1)\pi}{2} + \delta)|\sin(\frac{(2n+1)\pi}{2} + \delta)| \to \infty$ as $n \to \infty$ which implies there is no $\delta > 0$ that will keep $|h(x + \delta) - h(x)| < \varepsilon$ for all $x$. So, $h$ cannot be uniformly continuous.

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4. Prove: For all $x, y \leq 0$, we have $|e^x - e^y| \leq |x - y|$.

*Solution:* Fix $y \leq 0$ and consider the function $g(x) = e^x$ for $x \leq 0$. Apply the mean value theorem to the function $g$:

$$e^x - e^y = e^{\xi}(x - y),$$

where $\xi$ is between $x$ and $y$. Now, since $x$ and $y \leq 0$, we have $e^{\xi} \leq 1$; consequently,

$$|e^x - e^y| = |e^{\xi}(x - y)| \leq |x - y|$$

and the result follows.

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5. Suppose $f : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ has a continuous derivative which satisfies $f'(x) \geq m > 0$ for all $x \in [a, b]$. Show that

$$\left| \int_a^b \cos(f(x))\, dx \right| \leq \frac{2}{m}.$$

*Solution:* Since $-\frac{\pi}{2} \leq f(x) \leq \frac{\pi}{2}$ for all real $x$, it follows that $\cos(f(x)) \geq 0$ for all real $x$. Consequently,

$$\begin{aligned} \int_a^b \cos(f(x))\, dx &= \int_a^b \frac{\cos(f(x))f'(x)}{f'(x)}\, dx \\ &\leq \int_a^b \frac{\cos(f(x))f'(x)}{m}\, dx = \frac{1}{m}\int_a^b \cos(f(x))f'(x)\, dx \\ &= \frac{1}{m}\left(\sin(f(b)) - \sin(f(a))\right) \end{aligned}$$

from which it immediately follows that

$$\left| \int_a^b \cos(f(x))\, dx \right| \leq \frac{\left|\sin(f(b)) - \sin(f(a))\right|}{m} \leq \frac{2}{m}.$$

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Introductory Examination–Winter Session
Afternoon Exam–Probability

Tuesday, January 22, 2019

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

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3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Hölder's inequality for nonnegative random variables $X$ and $Y$ asserts that

$$\mathbb{E}(XY) \leq (\mathbb{E}\, X^p)^{1/p} \times (\mathbb{E}\, Y^q)^{1/q}$$

for any two real numbers $1 < p, q < \infty$ satisfying $(1/p) + (1/q) = 1$. Use Hölder's inequality to prove that if $Z$ is a nonnegative random variable and $0 < r < s < \infty$, then

$$(\mathbb{E}\, Z^r)^{1/r} \leq (\mathbb{E}\, Z^s)^{1/s}.$$

*Solution:* Apply Hölder's inequality with $X = Z^r$, $Y = 1$, $p = s/r$, and $q = s/(s-r)$ to find

$$\mathbb{E}\, Z^r \leq (\mathbb{E}\, Z^s)^{r/s} \times (\mathbb{E}\, 1^{s/(s-r)})^{(s-r)/s} = (\mathbb{E}\, Z^s)^{r/s}.$$

Take $r$th roots to complete the proof.

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2. Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ be independent identically distributed random variables which take the values $+1$ and $-1$ each with probability $\frac{1}{2}$. Define two sequences of random variables by

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

and

$$T_n = Y_1 + Y_2 + Y_3 + \cdots + Y_n$$

for positive integers $n \geq 1$.

Show that, for any integer $n \geq 1$,

$$\mathbb{P}[S_n = T_n] = \mathbb{P}[S_{2n} = 0].$$

*Solution:* Decompose the event $S_n = T_n$ into disjoint events of the form $S_n = T_n = k$. By independence of the $X$s and $Y$s, $S_n$ and $T_n$ are independent, so

$$\mathbb{P}[S_n = T_n = k] = \mathbb{P}[S_n = k, T_n = k] = \mathbb{P}[S_n = k]\mathbb{P}[T_n = k].$$

The probabilities $\mathbb{P}[S_n = k]$ and $\mathbb{P}[T_n = k]$ are Binomial$(n, \frac{1}{2})$ probabilities, so by summing over all possibilities, we have

$$\mathbb{P}[S_n = T_n] = \sum_{k=0}^{n} \left\{ \binom{n}{k} \left(\frac{1}{2}\right)^n \right\}^2 = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k}^2 = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}.$$

However, by the combinatorial identity

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

it then follows

$$\mathbb{P}[S_n = T_n] = \binom{2n}{n}\frac{1}{2^{2n}} = \mathbb{P}[S_{2n} = 0].$$

*Alternative approach*:

For each integer $i$, $1 \le i \le n$, the random variable $Y_i$ has the same distribution as $-X_{n+i}$. Consequently, $T_n$ has the same distribution as $-\sum_{j=n+1}^{2n} X_j = -S_{2n} + S_n$. It then follows that $P[S_n = T_n] = P[S_n = -S_{2n} + S_n] = P[S_{2n} = 0]$.

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3. There are four cards, two with the number 0 and two with the number 1. The cards are shuffled and the top two cards are put in one pile and the bottom two in another. Then one card is drawn at random from each pile and the average of numbers $\overline{X}$ on the drawn cards is computed.

   (a) Conditionally given that the piles are homogeneous, i.e., the same numbers appear within piles, what is the variance of $\overline{X}$ ?

   (b) Conditionally given that the piles are heterogeneous, i.e., the piles each contain two distinct numbers what is the variance of $\overline{X}$?

   (c) What is the variance of $\overline{X}$ unconditionally?

*Solution:* (a) Given pile homogeneity we always draw 0 and 1, so $\overline{X} \equiv 1/2$ and the conditional variance is zero.

(b) Given pile heterogeneity $\overline{X}$ is an average of two independent Bernoulli(1/2) random variables, so the conditional variance is $(1/2)(1-1/2)/2 = 1/8$.

(c) Pile homogeneity occurs with probability 1/3 so the expected value of the conditional variance is

$$1/3 \times 0 + 2/3 \times 1/8 = 1/12.$$

On the other hand, the expected value conditioned on the pile status is always 1/2, so the variance of the conditional expected value of $\overline{X}$ is zero. Consequently, the variance of $\overline{X}$ is 1/12, which by the way coincides with the variance of the sample mean for a simple random sample of size $n = 2$ (sampling without replacement) from a population of size $N = 4$ when the population variance is $\sigma^2 = p(1-p) = 1/4$.

$$[p(1-p)/n] \times [(N-n)/(N-1)] = (1/8) \times (2/3) = 1/12.$$

4. A coin is selected from a set of coins having head probability $p$ uniformly distributed on the interval $(0, 1)$. The selected coin is then flipped independently and repeatedly until the first head occurs. Let $X$ represent the trial index of this first head. Let $x \geq 1$ be an integer. Compute $\mathbb{P}(X = x)$.

*Solution:* We are told $p \sim \text{Uniform}(0, 1)$ and $X|p \sim \text{Geometric}(p)$. Therefore,

$$\mathbb{P}(X = x|p) = p(1 - p)^{x-1}.$$

Employ the law of total probability (and recognize the form of the Beta density):

$$\mathbb{P}(X = x) = \int_0^1 p(1 - p)^{x-1} \, dp = \frac{\Gamma(2)\Gamma(x)}{\Gamma(2 + x)} = \frac{(x - 1)!}{(x + 1)!} = \frac{1}{x(x + 1)}.$$

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5. Consider independent random variables $X$ and $Y$ each exponentially distributed, $X$ having parameter $\lambda_1$ and $Y$ having parameter $\lambda_2$. Find the cdf of $X/Y$.

*Solution:* First approach:

Since $\mathbb{P}(X > x) = e^{-\lambda_1 x}$, $\mathbb{P}(X > xY|Y) = e^{-\lambda_1 xY}$. Therefore, we have

$$\mathbb{P}(X > xY) = \mathbb{E}(\mathbb{P}(X > xY|Y)) = \mathbb{E}(e^{-\lambda_1 xY}) = \frac{\lambda_2}{\lambda_2 + \lambda_1 x},$$

where in the last equality we use the fact that $E(e^{tY}) = \frac{\lambda_2}{\lambda_2 - t}$ is the mgf of $Y$. Finally, for $x > 0$,

$$\mathbb{P}(X/Y \leq x) = 1 - \mathbb{P}(X/Y > x) = 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 x} = \frac{\lambda_1 x}{\lambda_2 + \lambda_1 x}.$$

*Alternative approach*:

Define $w = x/y$ and $v = y$. The inverse transformation is $x = wv$ and $y = v$ and the Jacobian determinant of this inverse transformation from $(w, v)$ to $(x, y)$ is

$$J = \det \begin{bmatrix} v & w \\ 0 & 1 \end{bmatrix} = v.$$

Consequently,

$$f_{W,V}(w,v) = f_{X,Y}(wv,v)|J| = \lambda_1\lambda_2 v e^{-\lambda_1 wv - \lambda_2 v} = \lambda_1\lambda_2 v e^{-(\lambda_1 w + \lambda_2)v}$$

giving the marginal

$$f_W(w) = \lambda_1\lambda_2 \int_0^\infty v e^{-(\lambda_1 w + \lambda_2)v}\, dv = \frac{\lambda_1\lambda_2}{(\lambda_1 w + \lambda_2)^2}.$$

Finally, for $x > 0$,

$$F_W(x) = P(W \leq x) = \int_0^x \frac{\lambda_1\lambda_2}{(\lambda_1 w + \lambda_2)^2}\, dw = \frac{\lambda_1 x}{\lambda_1 x + \lambda_2}.$$

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INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–LINEAR ALGEBRA

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7. **No calculators of any sort are needed or permitted.**

1. For two $n \times n$ matrices $A, B$, define $\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij}$. Let $U$ be an arbitrary $n \times n$ orthonormal matrix, i.e., the columns of $U$ form an orthonormal set of vectors. Show that $\langle A, B\rangle = \langle U^{-1}AU, U^{-1}BU\rangle$ for any two $n \times n$ matrices $A, B$. Hint: Relate $\langle A, B\rangle$ to the trace of $A^T B$.

*Solution:* Observe that $\langle A, B\rangle = \mathrm{tr}(A^T B)$. Thus,

$$\begin{aligned}
\langle A, B\rangle &= \mathrm{tr}(A^T B) \\
&= \mathrm{tr}(U^{-1}A^T BU) \\
&= \mathrm{tr}(U^{-1}A^T UU^{-1}BU) \\
&= \mathrm{tr}((U^{-1}AU)^T (U^{-1}BU)) \text{ Using the fact that } U^{-1} = U^T \text{ for orthonormal matrices} \\
&= \langle U^{-1}AU, U^{-1}BU\rangle.
\end{aligned}$$

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2. Let $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ be two vectors in $\mathbb{R}^3$. Show *directly* that the length of the cross product of $a$ and $b$ is at most the product of the lengths of each, i.e., $\|a \times b\| \leq \|a\|\|b\|$, where $\|(\ell_1, \ell_2, \ell_3)\| = \sqrt{\ell_1^2 + \ell_2^2 + \ell_3^2}$.

*Solution:* It suffices to verify $\|a\|^2\|b\|^2 - \|a \times b\|^2 \geq 0$. To this end

$$\begin{aligned}
\|a\|^2\|b\|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&= (a_1b_1)^2 + (a_1b_2)^2 + (a_1b_3)^2 + (a_2b_1)^2 + (a_2b_2)^2 + (a_2b_3)^2 \\
&\quad +(a_3b_1)^2 + (a_3b_2)^2 + (a_3b_3)^2
\end{aligned}$$

and

$$\begin{aligned}
\|a \times b\|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= (a_2b_3)^2 + (a_3b_2)^2 - 2a_2a_3b_2b_3 \\
&\quad +(a_3b_1)^2 + (a_1b_3)^2 - 2a_1a_3b_1b_3 \\
&\quad\quad +(a_1b_2)^2 + (a_2b_1)^2 - 2a_1a_2b_1b_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|a\|^2\|b\|^2 - \|a \times b\|^2 &= (a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2 + 2a_1a_2b_1b_2 + 2a_1a_3b_1b_3 + 2a_2a_3b_2b_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3)^2 \geq 0,
\end{aligned}$$

which was to be shown.

3. Suppose $\mathbf{P} = [\mathbf{A}\ \mathbf{B}]$ is an $n \times n$ orthogonal matrix. Show that $\mathbf{A}^T\mathbf{A}$ is an idempotent matrix. Recall a square matrix $C$ is idempotent provided $C^2 = C$.

*Solution:* Since $\mathbf{P}$ is orthogonal, $\mathbf{P}^T\mathbf{P} = \mathbf{I}_n$. However, this implies

$$\mathbf{P}^T\mathbf{P} = [\mathbf{A}\ \mathbf{B}]^T[\mathbf{A}\ \mathbf{B}] = \begin{bmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{bmatrix} [\mathbf{A}\ \mathbf{B}] = \begin{bmatrix} \mathbf{A}^T\mathbf{A} & \mathbf{A}^T\mathbf{B} \\ \mathbf{B}^T\mathbf{A} & \mathbf{B}^T\mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-m} \end{bmatrix}.$$

Therefore, $\mathbf{A}^T\mathbf{A} = \mathbf{I}_m$ is idempotent: $\mathbf{I}_m^2 = \mathbf{I}_m$. In fact, $\mathbf{B}^T\mathbf{B}$ is idempotent as well.

4. Suppose the $2 \times 2$ real symmetric matrix $A$ has eigenvalues satisfying $\lambda_1 > \lambda_2 > 0$. Show that the limit

$$\lim_{k\to\infty} \frac{A^k}{\lambda_1^k}$$

exists and find the value of this limit.

*Solution:* The spectral theorem applies to $A$: $A = PDP^T$, where $D = \mathrm{diag}(\lambda_1, \lambda_2)$ and $P = [v_1\ v_2]$ where $v_i$ is the (unit) eigenvector corresponding to $\lambda_i$, i.e., $\|v_i\|_2 = 1$. Then

$$\begin{aligned} \frac{A^k}{\lambda_1^k} &= \frac{\left([v_1\ v_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} [v_1\ v_2]^T\right)^k}{\lambda_1^k} \\ &= \frac{[v_1\ v_2] \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} [v_1\ v_2]^T}{\lambda_1^k} \\ &= [v_1\ v_2] \begin{bmatrix} 1 & 0 \\ 0 & (\frac{\lambda_2}{\lambda_1})^k \end{bmatrix} [v_1\ v_2]^T \end{aligned}$$

from which it is clear the limit as $k \to \infty$ exists (since $(\lambda_2/\lambda_1)^k \to 0$) and is

$$\lim_{k\to\infty} \frac{A^k}{\lambda_1^k} = [v_1\ v_2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [v_1\ v_2]^T = v_1 v_1^T.$$

5. If $A$ is a square $n$-by-$n$ matrix with $n \geq 2$, recall that the transposed matrix of cofactors of $A$ is called the *adjugate* or *classical adjoint* of $A$ and is denoted $\operatorname{adj} A$. For example, $\operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. In this problem you may use without proof the fact that

$$(\operatorname{adj} A)A = A(\operatorname{adj} A) = (\det A)I.$$

   (a) If $A$ is invertible, prove that $\operatorname{adj} A$ is also invertible and

$$\operatorname{adj} A = (\det A)A^{-1}.$$

   (b) If $\operatorname{rank} A \leq n-2$, prove that $\operatorname{adj} A = 0$.

   (c) If $\operatorname{rank} A = n-1$, prove that $\operatorname{rank} \operatorname{adj} A = 1$.

*Solution:*

(a) This is immediate from the stated fact, since if $A$ is invertible then $\det A \neq 0$.

(b) Every minor of $A$ of size $n-1$ vanishes, so (by its definition) the adjugate of $A$ vanishes.

(c) Some minor of $A$ of size $n-1$ is nonzero, so $\operatorname{adj} A \neq 0$ and $\operatorname{rank} \operatorname{adj} A \geq 1$. Moreover, some list of $n-1$ columns of $A$ is linearly independent, so the identity $(\operatorname{adj} A)A = (\det A)I = 0$ ensures that the null space of $\operatorname{adj} A$ has dimension at least $n-1$ and hence $\operatorname{rank} \operatorname{adj} A \leq 1$. We conclude that $\operatorname{rank} \operatorname{adj} A = 1$.

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