Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–REAL ANALYSIS

Tuesday, January 22, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and K a compact subset of the real line. Prove: $f(K) = \{f(x) \in \mathbb{R} : x \in K\}$ is compact.

2. Prove that there exists $\varepsilon > 0$ such that

$$\cos(\sin x) > \frac{1}{2}\left(1 + \cos^2(x)\right)$$

whenever $0 < |x| < \varepsilon$.

3. Let $f, g : [0, \infty) \to \mathbb{R}$ be uniformly continuous functions. Must their product be uniformly continuous? If so, prove it. If not, give a counterexample.

4. Prove: For all $x, y \leq 0$, we have $|e^x - e^y| \leq |x - y|$.

5. Suppose $f : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ has a continuous derivative which satisfies $f'(x) \geq m > 0$ for all $x \in [a, b]$. Show that

$$\left| \int_a^b \cos(f(x))\, dx \right| \leq \frac{2}{m}.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
AFTERNOON EXAM–PROBABILITY

Tuesday, January 22, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Hölder's inequality for nonnegative random variables X and Y asserts that

$$\mathbb{E}(XY) \le (\mathbb{E}\, X^p)^{1/p} \times (\mathbb{E}\, Y^q)^{1/q}$$

for any two real numbers $1 < p, q < \infty$ satisfying $(1/p) + (1/q) = 1$. Use Hölder's inequality to prove that if Z is a nonnegative random variable and $0 < r < s < \infty$, then

$$(\mathbb{E}\, Z^r)^{1/r} \le (\mathbb{E}\, Z^s)^{1/s}.$$

2. Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ be independent identically distributed random variables which take the values $+1$ and -1 each with probability $\frac{1}{2}$. Define two sequences of random variables by

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

and

$$T_n = Y_1 + Y_2 + Y_3 + \cdots + Y_n$$

for positive integers $n \ge 1$.

Show that, for any integer $n \ge 1$,

$$\mathbb{P}[S_n = T_n] = \mathbb{P}[S_{2n} = 0].$$

3. There are four cards, two with the number 0 and two with the number 1. The cards are shuffled and the top two cards are put in one pile and the bottom two in another. Then one card is drawn at random from each pile and the average of numbers $\overline{X}$ on the drawn cards is computed.

   (a) Conditionally given that the piles are homogeneous, i.e., the same numbers appear within piles, what is the variance of $\overline{X}$?

   (b) Conditionally given that the piles are heterogeneous, i.e., the piles each contain two distinct numbers, what is the variance of $\overline{X}$?

   (c) What is the variance of $\overline{X}$ unconditionally?

4. A coin is selected from a set of coins having head probability p uniformly distributed on the interval $(0, 1)$. The selected coin is then flipped independently and repeatedly until the first head occurs. Let X represent the trial index of this first head. Let $x \ge 1$ be an integer. Compute $\mathbb{P}(X = x)$.

5. Consider independent random variables X and Y each exponentially distributed, X having parameter λ_1 and Y having parameter λ_2. Find the cdf of X/Y.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Linear Algebra

Wednesday, January 23, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. For two $n \times n$ matrices A, B, define $\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij}$. Let U be an arbitrary $n \times n$ orthonormal matrix, i.e., the columns of U form an orthonormal set of vectors. Show that $\langle A, B\rangle = \langle U^{-1}AU, U^{-1}BU\rangle$ for any two $n \times n$ matrices A, B. Hint: Relate $\langle A, B\rangle$ to the trace of $A^T B$.

2. Let $a = (a_1, a_2, a_3)^T$ and $b = (b_1, b_2, b_3)^T$ be two vectors in $\mathbb{R}^3$. Show *directly* that the length of the cross product of a and b is at most the product of the lengths of each, i.e., $\|a \times b\| \leq \|a\|\|b\|$, where $\|(\ell_1, \ell_2, \ell_3)\| = \sqrt{\ell_1^2 + \ell_2^2 + \ell_3^2}$.

3. Suppose $\mathbf{P} = [\mathbf{A}\ \mathbf{B}]$ is an $n \times n$ orthogonal matrix. Show that $\mathbf{A}^T\mathbf{A}$ is an idempotent matrix. Recall a square matrix C is idempotent provided $C^2 = C$.

4. Suppose the 2×2 real symmetric matrix A has eigenvalues satisfying $\lambda_1 > \lambda_2 > 0$. Show that the limit
$$\lim_{k\to\infty} \frac{A^k}{\lambda_1^k}$$
exists and find the value of this limit.

5. If A is a square n-by-n matrix with $n \geq 2$, recall that the transposed matrix of cofactors of A is called the *adjugate* or *classical adjoint* of A and is denoted $\operatorname{adj} A$. For example, $\operatorname{adj} \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. In this problem you may use without proof the fact that
$$(\operatorname{adj} A)A = A(\operatorname{adj} A) = (\det A)I.$$
   (a) If A is invertible, prove that $\operatorname{adj} A$ is also invertible and
   $$\operatorname{adj} A = (\det A)A^{-1}.$$
   (b) If $\operatorname{rank} A \leq n - 2$, prove that $\operatorname{adj} A = 0$.
   (c) If $\operatorname{rank} A = n - 1$, prove that $\operatorname{rank} \operatorname{adj} A = 1$.