Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Real Analysis

Tuesday, January 21, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. For a function $f : X \to Y$ and $A \subseteq X$ we define the image of A under f to be the set $f[A] := \{y \in Y : y = f(x) \text{ for some } x \in A\}$. Prove the following statements are equivalent:

   (a) f is one to one.

   (b) $f[A \cap B] = f[A] \cap f[B]$ holds for every two subsets A, B of X.

Solution: (b) $\implies$ (a): If $x_1 \neq x_2$, then

$$\{f(x_1)\} \cap \{f(x_2)\} = f[\{x_1\}] \cap f[\{x_2\}] = f[\{x_1\} \cap \{x_2\}] = f[\emptyset] = \emptyset,$$

so $f(x_1) \neq f(x_2)$.

(a) $\implies$ (b): Let $f : X \to Y$ be one-to-one and suppose A and B are any two subsets of X.

Case 1: If $f[A] \cap f[B] = \varnothing$, then $A \cap B = \varnothing$ and it follows $f[A \cap B] = f[A] \cap f[B]$.

Case 2: If $f[A] \cap f[B] \neq \varnothing$, then consider any $y \in f[A] \cap f[B]$. Since $y \in f[A]$, there is an $x_1 \in A$ such that $y = f(x_1)$. Similarly, there is an $x_2 \in B$ such that $y = f(x_2)$. f being one-to-one implies $x_1 = x_2 = x$ and consequently $x \in A \cap B$. Therefore, $y \in f[A \cap B]$. This shows $f[A \cap B] \supseteq f[A] \cap f[B]$. The other inclusion is immediate since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ implies $f[A \cap B] \subseteq f[A] \cap f[B]$.

---

2. Determine whether the following integral is convergent or divergent. If it is convergent, evaluate it:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$$

Solution: The integral converges. Using polar coordinates: $dx\, dy = r\, dr\, d\theta$ and $x^2 + y^2 = r^2$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx\, dy}{(1 + x^2 + y^2)^{\frac{3}{2}}} = \int_0^{2\pi} \int_0^{\infty} \frac{r\, dr\, d\theta}{(1 + r^2)^{\frac{3}{2}}} = \frac{1}{2} \int_0^{2\pi} \int_1^{\infty} \frac{du\, d\theta}{u^{\frac{3}{2}}} = 2\pi.$$

---

3. If (x_n) is a real-valued sequence and $x > 0$, is it true that $x_n \to x$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \frac{x_n - x}{x_n + x} = 0?$$

Prove your answer.

Solution: The answer is yes. If $x_n \to x$, then $(x_n - x)/(x_n + x) \to (x - x)/(x + x) = 0$. Conversely, suppose that $x_n \not\to x$. Then (for example, consider the lim inf or lim sup of the sequence) there exists $\lambda \in [-\infty, \infty]$ with $\lambda \neq x$ and a sequence (n_k) such that $x_{n_k} \to \lambda$. If $\lambda = \pm\infty$, then $(x_{n_k} - x)/(x_{n_k} + x) \to 1$ and so $(x_n - x)/(x_n + x) \not\to 0$. If λ is finite and different from $-x$, then $(x_{n_k} - x)/(x_{n_k} + x) \to (\lambda - x)/(\lambda + x) \neq 0$ and so again $(x_n - x)/(x_n + x) \not\to 0$. Finally, if $\lambda = -x$, then $|(x_{n_k} - x)/(x_{n_k} + x)| \to \infty \neq 0$ and so once again $(x_n - x)/(x_n + x) \not\to 0$.

---

4. Let f, g be differentiable real-valued functions defined on a fixed open interval I of the real line. Suppose $c \in I$ is such that

   (i) $f(c) = g(c) = 0$,

   (ii) $g(x) \neq 0$ for all $x \in I$, with $x \neq c$, and

   (iii) $g'(c) \neq 0$.

   Prove this version of the L'Hôpital rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$.

Solution: For $x \in I$, $x \neq c$, $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(c)}{x - c} \Big/ \dfrac{g(x) - g(c)}{x - c}$. Since $\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$ and $\lim_{x\to c} \dfrac{g(x) - g(c)}{x - c} = g'(c)$, we have by ordinary limit rules

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)}.$$

---

5. For each positive integer n, define $f_n : (0, \infty) \to \mathbb{R}$ by $f_n(x) := 1/(nx)$. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) := 0$. Prove that as $n \to \infty$ we have

   (a) $f_n \to f$ (pointwise) on $(0, \infty)$;

(b) f_n does *not* converge uniformly to f on $(0, \infty)$; and

(c) for any $c > 0$, the sequence f_n converges uniformly to f on $[c, \infty)$.

Solution:

(a) This is obvious. To spell out the obvious, given $x \in (0, \infty)$ and $\epsilon > 0$, for every integer $n > 1/(\epsilon x)$ we have $|f_n(x) - f(x)| = f_n(x) = 1/(nx) < \epsilon$.

(b) If (for example) $x_n := 1/n \in (0, \infty)$, then $|f_n(x_n) - f(x_n)| = f_n(x_n) = 1 \not\to 0$.

(c) Let $\|\cdot\|$ denote sup-norm for functions defined on the domain $[c, \infty)$. Then

$$\begin{aligned} \|f_n - f\| &= \sup\{|f_n(x) - f(x)| : x \in [c, \infty)\} = \sup\{1/(nx) : x \in [c, \infty)\} \\ &= 1/(nc) \to 0. \end{aligned}$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Afternoon Exam–Probability

Tuesday, January 21, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. The amount of time that a certain type of component functions before failing is a random variable with probability density function

$$f(x) = 2x, \quad 0 < x < 1.$$

Once the component fails it is immediately replaced by another of the same type. Let X_i denote the lifetime of the ith component to be put in use, and assume that $X_1, X_2, \ldots$ are independent. The long-term rate at which failures occur, call it R, is defined by

$$R := \lim_{n\to\infty} \frac{n}{S_n}$$

where S_n is the time at which the nth failure occurs. What is the distribution of the random variable R?

Solution: This is an exercise from Chapter 8 in Ross. First observe that f is indeed a probability density function, since

$$\int f(x)\,dx = \int_0^1 2x\,dx = 1.$$

From the description given we have $S_n = \sum_{i=1}^n X_i$. By the strong law of large numbers, $R = 1/\mu$ with probability one, where

$$\mu := \mathbb{E}\,X_1 = \int x f(x)\,dx = \int_0^1 2x^2\,dx = 2/3.$$

Thus $R = 3/2$ with probability one.

---

2. Suppose that X_1, X_2, X_3, $\ldots$ are i.i.d. random variables with mean 5 and standard deviation 2. Let N be a positive integer-valued random variable with mean 10 and standard deviation 3 which is independent of $\{X_i\}_{i=1}^{\infty}$

(a) Calculate the variance of the random sum of random variables

$$S_N = X_1 + X_2 + X_3 + \cdots + X_N.$$

(b) For comparison, calculate the variance of the sum of the fixed number of random variables

$$S_{10} = X_1 + X_2 + X_3 + \cdots + X_{10},$$

for which the number of summands is the mean of N.

Solution: (a) The overall plan: Because S_N is a sum of N random variables, it is natural to condition on the value of N. To calculate the variance while using conditioning, use the Law of Total Variance.

First, calculate

$$\begin{aligned}
&E[S_N|N = n] \\
&\qquad = E[X_1 + X_2 + X_3 + \cdots + X_N|N = n] \\
&\qquad = E[X_1 + X_2 + X_3 + \cdots + X_n|N = n] \\
&\qquad \text{but since the } \{X_i\}_{i=1}^{\infty} \text{ and } N \text{ are independent,} \\
&\qquad = E[X_1 + X_2 + X_3 + \cdots + X_n] \\
&\qquad \text{so by linearity of expectation and identical distribution} \\
&\qquad = 5n.
\end{aligned}$$

Thus,

$$E[S_N|N] = 5N.$$

Similarly,

$$\text{Var}(S_N|N) = \text{Var}(X_1)N = 4N.$$

By the Law of Total Variance,

$$\begin{aligned}
\text{Var}(S_N) &= E[\text{Var}(S_N|N)] + \text{Var}(E[S_N|N]) \\
&= E[4N] + \text{Var}(5N) \\
&= 4E[N] + 25\text{Var}(N) \\
&= 4(10) + 25(9) \\
&= 265.
\end{aligned}$$

(b) By independence, the variance of the sum is the sum of the variances, so by identical distribution,

$$\text{Var}(S_n) = 10\text{Var}(X_1) = 10(4) = 40.$$

Thus, we see that the variance of the random sum is almost seven times that of the fixed sum with the same average number of terms.

3. Consider a random variable X having a probability density function of the form $f(x) = cx^2$ for $-1 \le x \le 1$, for some constant c. Find c and the probability density function of $Y = X^2$.

Solution: Since $\int_{-1}^{1} cx^2\, dx = 1$ we have $c = \frac{1}{\int_{-1}^{1} x^2\, dx} = \frac{3}{2}$. Now, since $-1 \le X \le 1$, we have $0 \le Y \le 1$. If $y \le 0$, $P(Y \le y) = 0$; also, if $y \ge 1$, $P(Y \le y) = 1$.

If $0 < y < 1$, $P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{3}{2} \int_{-\sqrt{y}}^{\sqrt{y}} x^2\, dx = y^{3/2}$.

Thus, we shown the cumulative distribution function of Y is

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ y^{3/2} & 0 < y < 1 \\ 1 & y \ge 1 \end{cases}.$$

Consequently, $f_Y(y) = \frac{3\sqrt{y}}{2}$ for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.

---

4. Let X and Y be independent random variables with the standard Normal$(0, 1)$ distribution. Find the conditional distribution of X given $X = Y$.

Solution: Let $W = X - Y$ so that W is Normal$(0, 2)$, i.e., W has the pdf $f_W(w) = \frac{e^{-w^2/4}}{2\sqrt{\pi}}$ for real w. It follows that

$$f_{X,W}(x, w) = f_{X,Y}(x, x - w) = \varphi(x)\varphi(x - w),$$

where $\varphi(x)$ represents the standard normal pdf: $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ for real x.

Therefore,

$$f_{X|W}(x|0) = \frac{f_{X,W}(x, 0)}{f_W(0)} = \frac{\varphi(x)^2}{\frac{1}{2\sqrt{\pi}}} = \frac{e^{-x^2}}{\sqrt{\pi}},$$

which shows the conditional distribution of X given $X - Y = 0$ is Normal$(0, \frac{1}{2})$.

---

5. In a book published by Yale University press, it was stated that more people know who said "What's up, Doc?" than who said "Give me liberty or give me death". Let's say at Johns Hopkins it's the other way around: say

- 72% know who said both
- 18% know who said "Give me liberty or give me death" and do not know who said "What's up, Doc?"
- 8% know who said "What's up, Doc?" and do not know who said "Give me liberty or give me death"
- 2% know neither.

If you took a random sample of 100 Hopkins students, what is the approximate chance that in the sample more students know who said "What's Up, Doc?" than "Give me liberty or give me death"?

You may leave your answer in terms of the standard normal cumulative distribution function Φ.

Solution: For $i = 1, 2, \ldots, 100$, let $X_i = 1$ if the i-th person knows who said "What's up, Doc?" and $X_i = 0$ otherwise; similarly, let $Y_i = 1$ if the i-th person knows who said "Give me liberty or give me death" and $Y_i = 0$ otherwise. Now,

$$\begin{aligned} E(X_i - Y_i) &= .80 - .90 = -.10 \\ \mathrm{Var}(X_i - Y_i) &= \mathrm{Var}(X_i) + \mathrm{Var}(Y_i) - 2\mathrm{Cov}(X_i, Y_i) \\ &= .80(.20) + .90(.10) - 2[.72 - .80(.90)] = .25, \end{aligned}$$

and, therefore, $\sum_{i=1}^{100}(X_i - Y_i)$ has mean $100(-.10) = -10$ and variance $100(.25) = 25$. Thus, by the central limit theorem

$P(\sum_{i=1}^{100}(X_i - Y_i) > 0) = P\left(\frac{\sum_{i=1}^{100}(X_i - Y_i) - (-10)}{\sqrt{25}} > \frac{10}{5}\right) \approx 1 - \Phi(2) \approx .0228.$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–WINTER SESSION
MORNING EXAM–LINEAR ALGEBRA

Wednesday, January 22, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let u and v be distinct eigenvectors of a matrix A such that $u+v$ is also an eigenvector of A. Prove or disprove: $u-v$ is necessarily an eigenvector of A.

Solution: Let λ_1, λ_2, and λ_3 be the eigenvalues associated with u, v, and $u+v$, respectively. If u and v are linearly dependent, then $\lambda_1 = \lambda_2 = \lambda_3$. If u and v are linearly independent, then $\lambda_1 u + \lambda_2 v = \lambda_3(u+v)$, so $(\lambda_1 - \lambda_3)u + (\lambda_2 - \lambda_3)v = 0$, which implies that $\lambda_1 = \lambda_2 = \lambda_3$ (using the linear independence of u and v here). In either case, the eigenvectors u, v and $u+v$ are associated with the same eigenvalue λ. Thus, $A(u-v) = \lambda u - \lambda v = \lambda(u-v)$. Furthermore, we note that u and v are distinct so that $u-v$ is nonzero, therefore $u-v$ is an eigenvector of A.

---

2. Let V_1 and V_2 be two proper subspaces of a vector space V. Show that there exists a vector $v \in V$ such that $v \notin V_1$ and $v \notin V_2$.

Solution: Without loss of generality we can assume $V_1 \neq V_2$, otherwise the result is immediate. Since V_1 is a proper subspace of V, there exists $w_1 \notin V_1$. If in addition $w_1 \notin V_2$, we are done. If $w_1 \in V_2$, since V_2 is proper, there exists vector $w_2 \notin V_2$. If in addition $w_2 \notin V_1$, we are done. Otherwise, now we have

$$w_1 \notin V_1,\ w_2 \in V_1,\ w_1 \in V_2,\ w_2 \notin V_2$$

Letting $v = w_1 + w_2$, we know that $v \notin V_1$ and $v \notin V_2$.

Alternate solution: By contradiction, if one assumes that $V_1 \cup V_2 = V$, this implies that one of the subspaces is included in the other. Indeed, otherwise, we could find $x \in V_1 \cap V_2^c$ and $y \in V_2 \cap V_1^c$, in which case $x + y \notin V_1 \cup V_2 = V$, which is impossible. But then we would get that either V_1 or V_2 is equal to V which is a contradiction.

---

3. Let A and B be any $m \times n$ matrices, show that

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

Solution: We have

$$\operatorname{Range}(A+B) \subset \operatorname{Range}(A) + \operatorname{Range}(B).$$

Hence

$$\dim(\mathrm{Range}(A+B)) \leq \dim(\mathrm{Range}(A)+\mathrm{Range}(B)) \leq \dim(\mathrm{Range}(A))+\dim(\mathrm{Range}(B)).$$

---

4. Suppose A and B are $n \times n$ matrices. Show $AB - BA$ cannot be a nonzero multiple of the identity matrix.

   Hint: Consider traces.

   Solution:
   $$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0.$$
   So if $AB - BA = cI_n$ then
   $$0 = \mathrm{tr}(cI_n) = nc.$$
   So $c = 0$.

---

5. A matrix square matrix A is called *idempotent* if $A^2 = A$. Suppose P and Q are $n \times n$ idempotent matrices and that $P + Q$ is also idempotent. Prove that $PQ = 0$.

   Solution: Observe that $(P+Q)(P+Q) = P+Q$ gives $P+Q+PQ+QP = P+Q$ so $PQ = -QP$.

   Then we have
   $$PQPQ = P(QP)Q = P(-PQ)Q = -P^2Q^2 = -PQ.$$
   On the other hand, we have
   $$PQPQ = (-QP)PQ = -QPQ = -(QP)Q = -(-PQ)Q = PQ,$$
   so we conclude that $PQ = PQPQ = -PQ$, hence $PQ = 0$.

   Alternate solution. After showing $PQ = -QP$, we can left multiply by P to get $P^2Q = -PQP = QP^2$ and thus $PQ = QP$. Together, $PQ = -QP = QP$ implies $PQ = 0$.
