Department of Applied Mathematics and Statistics
The Johns Hopkins University

# Introductory Examination–Winter Session
## Morning Exam–Real Analysis

Tuesday, January 23, 2018

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let $E$ be a compact set and $F \subseteq E$ a closed subset of $E$. Prove that $F$ is compact by an open covering argument.

*Solution:* If $F = E$ then there is nothing to show. Assume $F$ is a proper subset of $E$ and let $(\mathcal{O}_\alpha)$ be any collection of open subsets of $E$ whose union covers $F$, i.e., $F \subseteq \bigcup_\alpha \mathcal{O}_\alpha$. Since $F^c$ is also open, $F^c$ together with $(\mathcal{O}_\alpha)$ is an open cover of $E$. $E$ is compact so there is a finite subcollection of these sets which covers $E$ and, consequently, also $F$.

If $F^c$ is an element of this finite subcollection, discard it. The result is the desired finite subcollection of $(\mathcal{O}_\alpha)$ that covers $F$.

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2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere, and suppose that $f'(x) \neq 0$ for every $x \in \mathbb{R}$. Show that $f$ is one-to-one (injective).

*Solution:* Suppose to the contrary that $f(a) = f(b)$ for $a \neq b$. By the Mean Value Theorem, we obtain that there exists $c \in (a, b)$ such that $f'(c) = 0$, contradicting the hypothesis.

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3. Compute

$$\int_0^1 \int_0^1 \frac{\min\{x,y\}}{\max\{x,y\}}\, dx\, dy.$$

*Solution:*

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{\min\{x,y\}}{\max\{x,y\}}\, dx\, dy &= \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_0^1 \int_y^1 \frac{y}{x}\, dx\, dy \\
&= \int_0^1 \int_0^y \frac{x}{y}\, dx\, dy + \int_0^1 \int_0^x \frac{y}{x}\, dy\, dx \\
&= 2\int_0^1 \int_0^y \frac{x}{y}\, dx\, dy = 2\int_0^1 \frac{y}{2}\, dy = \frac{1}{2}.
\end{aligned}$$

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4. Consider the sequence of functions $f_n(x) = \cos(nx)$ for $n = 1, 2, \ldots$. Prove that there is no subsequence that converges uniformly on $\mathbb{R}$.

*Solution:* Proof by contradiction. A uniformly convergent sequence of continuous functions has a continuous limit. The functions $f_n$ are continuous, so if the sequence has a uniformly convergent subsequence $f_{n_k}$, that subsequence would converge uniformly to some continuous function $g$.

In particular, since $f_{n_k}(0) = 1$ for all $n$ it must be the case that $g(0) = 1$, and by continuity, there exists $\delta > 0$ such that $g(x) > 1/2$ for $|x| < \delta$. By uniform convergence, taking $\epsilon = 1/4$ there exists $N$ such that $|f_{n_k}(x) - g(x)| \leq \epsilon = 1/4$ for all $x$ provided that $n_k > N$.

From this we see that

$$f_{n_k}(x) \geq g(x) - \epsilon > 1/2 - 1/4 = 1/4$$

whenever $|x| < \delta$ and $n_k > N$.

On the other hand, we can choose $n_k > N$ such that $\pi/n_k < \delta$ so by defining $x = \pi/n_k$ we have $|x| < \delta$ but $f_{n_k}(x) = \cos(\pi) = -1$, which is a contradiction.

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5. Is the integral $\int_0^1 (\sin x + \cos x)^{1/x}\, dx$ convergent?

*Solution:* The integrand is well defined and continuous on $(0, 1]$ because $\sin x + \cos x > 0$ on $(0, \pi/2] \supset (0, 1]$. It is therefore sufficient to show that the limit $\lim_{x \to 0} (\sin x + \cos x)^{1/x}$ exists.

We have $\sin x = x + o(x^2)$ at $x = 0$ and $\cos x = 1 + o(x)$, so that (taking logs)

$$\frac{1}{x}\log(\sin x + \cos x) = \frac{1}{x}\log(1 + x + o(x)) = \frac{1}{x}(x + o(x))$$

which converges to 1 when $x \to 0$. Hence

$$\lim_{x \to 0} (\sin x + \cos x)^{1/x} = e$$

and the integral converges.

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Department of Applied Mathematics and Statistics
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INTRODUCTORY EXAMINATION–WINTER SESSION
AFTERNOON EXAM–PROBABILITY

Tuesday, January 23, 2018

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $X$ be a random variable with moment-generating function

$$M_X(t) = \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{3}e^{5t}.$$

Calculate the variance of $X$.

*Solution:* A slightly insightful approach: The definition of the moment-generating function is $M_X(t) = E[e^{tX}]$, which may be interpreted as a weighted average of exponential functions. Thus, for the function given, recognizing that it is a weighted average, we can see that the random variable $X$ takes the value 2 with probability $\frac{1}{2}$, the value 3 with probability $\frac{1}{6}$, and the value 5 with probability $\frac{1}{3}$. From this, we can calculate

$$E[X] = \frac{1}{2}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{3}\cdot 5 = \frac{19}{6},$$

and

$$E[X^2] = \frac{1}{2}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{3}\cdot 25 = \frac{71}{6},$$

so

$$Var(X) = \frac{71}{6} - \left(\frac{19}{6}\right)^2 = \frac{426 - 361}{36} = \frac{65}{36}.$$

The brute force approach: The first and second moments can be calculated by differentiation of the moment-generating function:

$$E[X] = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{1}{2}2e^{2t} + \frac{1}{6}3e^{3t} + \frac{1}{3}5e^{5t}\right]_{t=0}$$
$$= \frac{1}{2}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{3}\cdot 5 = \frac{19}{6}$$

and

$$E[X^2] = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{1}{2}4e^{2t} + \frac{1}{6}9e^{3t} + \frac{1}{3}25e^{5t}\right]_{t=0}$$
$$\frac{1}{2}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{3}\cdot 25 = \frac{71}{6},$$

as found above, and continue to calculate the variance.

2. Let $X$ be a continuous random variable with a continuous probability density function $f(x)$, such that $E(|X|) < \infty$. Determine with justification the value of $\theta$ that minimizes $E[|X - \theta|]$.

*Solution:* First

$$L(\theta) = E[|X-\theta|] = \int_{-\infty}^{\infty} |x-\theta| f(x)\,dx$$
$$= \int_{-\infty}^{\theta} (\theta - x) f(x)\,dx + \int_{\theta}^{\infty} (x-\theta) f(x)\,dx.$$

Differentiating $L(\theta)$ with respect to $\theta$ (using the Liebniz rule) and equating to zero:

$$L'(\theta) = \int_{-\infty}^{\theta} f(x)\,dx - \int_{\theta}^{\infty} f(x)\,dx = 0$$

implies

$$\int_{-\infty}^{\theta} f(x)\,dx = \int_{\theta}^{\infty} f(x)\,dx$$

and since $f$ is a pdf, $\int_{-\infty}^{\theta} f(x)\,dx = \frac{1}{2}$, i.e., $\theta$ is the *median*.
It is clear that the median minimizes $E[|X - \theta|]$ since $L''(\theta) = 2f(\theta) \geq 0$ shows $L(\theta)$ is convex.

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3. A room contains $m$ married couples. Two people are selected uniformly at random from the $2m$ total people to form a pair, then two other people are uniformly selected from the remaining $2m - 2$ people to form another pair, and so on until all $2m$ people are paired. Let $M$ be the random variable that counts the number of married couples paired. Compute $E(M)$.

*Solution:* Let $M_i = \begin{cases} 1 & \text{if married couple } i \text{ is paired} \\ 0 & \text{if not} \end{cases}$ so that $M = \sum_{i=1}^{m} M_i$.
$E(M_i) = P(\text{married couple } i \text{ is paired})$. Now, $P(\text{married couple } i \text{ is paired}) = \frac{1}{2m-1}$ since the first person in the couple must be matched with his/her one spouse out of the $2m - 1$ remaining people. Finally, $E(M) = \sum_{i=1}^{m} E(M_i) = \sum_{i=1}^{m} \frac{1}{2m-1} = \frac{m}{2m-1}$.

*Alternate solution*: Another way to see that $P(\text{married couple } i \text{ is paired}) = \frac{1}{2m-1}$ is as follows: the number of (unordered) pairings that have couple $i$ paired is in one-to-one correspondence with the number of (unordered) pairings of the remaining $2m - 2$

people, namely, $\binom{2m-2}{2,2,\ldots,2}/(m-1)!$ and therefore,

$$P(\text{married couple } i \text{ is paired}) = \frac{\binom{2m-2}{2,2,\ldots,2}/(m-1)!}{\binom{2m}{2,2,\ldots,2}/m!} = \frac{1}{2m-1}.$$

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4. Let $X$ be a random variable which takes values in the interval $[-M, M]$ only. Show that

$$P(|X| \geq a) \geq \frac{E(|X|) - a}{M - a},$$

if $0 \leq a < M$.

*Solution:*

$$\begin{aligned}
E(|X|) - a &= \int_\Omega (|X| - a)\, dP \\
&= \int_{\{|X| \geq a\}} (|X| - a)\, dP + \int_{\{|X| < a\}} (|X| - a)\, dP \\
&\leq \int_{\{|X| \geq a\}} (|X| - a)\, dP \\
&\leq \int_{\{|X| \geq a\}} (M - a)\, dP = (M - a) P(|X| \geq a).
\end{aligned}$$

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5. Suppose $X$ and $Y$ are independent random variables with the same exponential pdf $f(x) = e^{-x}$ for $x > 0$. Show that $X + Y$ and $X/Y$ are independent.

*Solution:* The joint pdf of $X, Y$ is $f_{X,Y}(x, y) = e^{-(x+y)}$ for $x > 0, y > 0$.

Let $u = x + y$ and $v = x/y$. Note $x > 0, y > 0$ implies $u > 0$ and $v > 0$. After some manipulation we find the inverse transformation to be

$$x = \frac{uv}{v+1} \quad \text{and} \quad y = \frac{u}{v+1}.$$

Now, we compute

$$\frac{\partial x}{\partial u} = \frac{v}{v+1}, \quad \frac{\partial x}{\partial v} = \frac{u}{(v+1)^2}, \quad \frac{\partial y}{\partial u} = \frac{1}{v+1}, \quad \text{and} \quad \frac{\partial y}{\partial v} = -\frac{u}{(v+1)^2}$$

implying the Jacobian determinant is $J = \frac{v}{v+1} \cdot (-\frac{u}{(v+1)^2}) - \frac{u}{(v+1)^2}(\frac{1}{v+1}) = \frac{-uv-u}{(v+1)^3} = -\frac{u}{(v+1)^2}$. The transformation $U = X + Y$ and $V = X/Y$ induces the pdf of $U, V$:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x,y)|J| = f_{X,Y}(\frac{uv}{v+1}, \frac{u}{v+1})\left|-\frac{u}{(v+1)^2}\right| \\ &= \frac{u}{(v+1)^2}e^{-(\frac{uv}{v+1}+\frac{u}{v+1})} \\ &= ue^{-u} \cdot \frac{1}{(v+1)^2} \quad \text{for } u > 0, v > 0. \end{aligned}$$

A couple of simple integrations show that the marginal pdfs of $U$ and of $V$ are:

$$f_U(u) = ue^{-u} \text{ for } u > 0 \text{ and } f_V(v) = \frac{1}{(v+1)^2} \text{ for } v > 0.$$

Therefore, since the joint pdf of $U, V$ factors as the product of its marginal pdfs, we've shown $X + Y$ and $X/Y$ to be independent!

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Department of Applied Mathematics and Statistics
The Johns Hopkins University

# Introductory Examination–Winter Session
## Morning Exam–Linear Algebra

Wednesday, January 24, 2018

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

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5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

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7. **No calculators of any sort are needed or permitted.**

1. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove this:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{1}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)}.$$

*Solution:* Let $\mathbf{v}$ be the vector $[a_1\ a_2\ \cdots\ a_n]^t$ and let $\mathbf{1}$ be the $n$-vector of all 1s. The Cauchy-Schwarz inequality gives

$$\mathbf{v} \cdot \mathbf{1} \leq \|\mathbf{v}\| \cdot \|\mathbf{1}\|.$$

This expands to

$$a_1 + a_2 + \cdots + a_n \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \times \sqrt{n}$$

and the result follows by dividing both sides by $n$.

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2. Do there exist polynomials $p(x)$, $q(x)$, $r(y)$, $s(y)$ such that

$$1 + xy + x^2y^2 = p(x)r(y) + q(x)s(y)$$

holds identically for all $x, y$? Please justify your assertion.

*Solution:* If such polynomials exist, then setting $y = -1, 0, 1$ shows that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are all in the linear span of $p(x)$ and $q(x)$. However, $\{1 - x + x^2, 1, 1 + x + x^2\}$ is a linearly independent set, so it cannot be contained in a two-dimensional span. This contradiction implies that the relation cannot hold.

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3. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear function, i.e., $L(x + y) = L(x) + L(y)$ for all $x, y \in \mathbb{R}^n$ and $L(\lambda x) = \lambda L(x)$. Show that there exists $a \in \mathbb{R}^n$ such that $L(x) = \sum_{i=1}^n a_i x_i$ for all $x \in \mathbb{R}^n$.

*Solution:* For each $i = 1, \ldots, n$, define $a_i = L(e^i)$, where $e^i$ is the $i$-th standard unit vector. Then $L(x) = L(\sum_i x_i e^i) = \sum_i x_i L(e^i) = \sum_i x_i a_i$.

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4. Let $S$ be an $n$ by $n$ real matrix with rank $m$. Prove that there exist two real matrices $A$ and $B$ such that $A$ is $n$ by $m$, $B$ is $m$ by $n$ and $S = AB$.

*Solution:* Note that one must have $m \leq n$. Let $e_1, \ldots, e_m \in \mathbb{R}^n$ be a basis of Range($S$). Take $A = [e_1, \ldots, e_m]$. Since $A$ has rank $m$, $A^T A$ is invertible, and if $B$ is such that $S = AB$, we must have $A^T S = A^T AB$, yielding

$$B = (A^T A)^{-1} A^T S$$

as the only possible choice. Taking this $B$, one then has $AB = S$. Indeed, if $x \in \mathbb{R}^n$, then $Sx \in$ Range($S$) is a linear combination of the columns of $A$, i.e., there exists $\lambda \in \mathbb{R}^m$ such that $Sx = A\lambda$. One then has

$$ABx = A(A^T A)^{-1} A^T Sx = A(A^T A)^{-1} A^T A\lambda = A\lambda = Sx.$$

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5. Suppose $A$ and $B$ are real $n \times n$ matrices. Prove that, if $A$ is similar to $B$, then $e^A$ is similar to $e^B$.

*Solution:* Since $A$ is similar to $B$, there is a nonsingular matrix $S$ such that $B = SAS^{-1}$. Also, $B^2 = (SAS^{-1})(SAS^{-1}) = SA^2S^{-1}$, and, generally, for any integer $k \geq 0$, $B^k = SA^kS^{-1}$. Therefore,

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = \sum_{k=0}^{\infty} \frac{1}{k!} SA^kS^{-1} = S\left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k\right) S^{-1} = Se^AS^{-1}$$

showing $e^A$ is similar to $e^B$ (by the same similarity transformation!).

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