Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session
Morning Exam–Real Analysis

Tuesday, January 23, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let E be a compact set and $F \subseteq E$ a closed subset of E. Prove that F is compact by an open covering argument.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere, and suppose that $f'(x) \neq 0$ for every $x \in \mathbb{R}$. Show that f is one to one (injective).

3. Compute

$$\int_0^1 \int_0^1 \frac{\min\{x,y\}}{\max\{x,y\}}\, dx\, dy.$$

4. Consider the sequence of functions $f_n(x) = \cos(nx)$ for $n = 1, 2, \ldots$. Prove that there is no subsequence that converges uniformly on $\mathbb{R}$.

5. Is the integral $\int_0^1 (\sin x + \cos x)^{\frac{1}{x}}\, dx$ well defined?

Department of Applied Mathematics and Statistics
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INTRODUCTORY EXAMINATION–WINTER SESSION
AFTERNOON EXAM–PROBABILITY

Tuesday, January 23, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let X be a random variable with moment-generating function

$$M_X(t) = \frac{1}{2}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{3}e^{5t}.$$

Calculate the variance of X.

2. Let X be a continuous random variable with a continuous probability density function $f(x)$. Determine with justification the value of θ that minimizes $E[|X - \theta|]$.

3. A room contains m married couples. Two people are selected uniformly at random from the $2m$ total people to form a pair, then two other people are uniformly selected from the remaining $2m - 2$ people to form another pair, and so on until all $2m$ people are paired. Let M be the random variable that counts the number of married couples paired. Compute $E(M)$.

4. Let X be a random variable which takes values in the interval $[-M, M]$ only. Show that

$$P(|X| \geq a) \geq \frac{E(|X|) - a}{M - a},$$

if $0 \leq a < M$.

5. Suppose X and Y are independent random variables with the same exponential pdf $f(x) = e^{-x}$ for $x > 0$. Show that $X + Y$ and X/Y are independent.

Department of Applied Mathematics and Statistics
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Introductory Examination–Winter Session
Morning Exam–Linear Algebra

Wednesday, January 24, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove this:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \leq \sqrt{\frac{1}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)}.$$

2. Do there exist polynomials $p(x)$, $q(x)$, $r(y)$, $s(y)$ such that

$$1 + xy + x^2y^2 = p(x)r(y) + q(x)s(y)$$

holds identically for all x, y? Please justify your assertion.

3. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a linear function, i.e., $L(x + y) = L(x) + L(y)$ for all $x, y \in \mathbb{R}^n$ and $L(\lambda x) = \lambda L(x)$. Show that there exists $a \in \mathbb{R}^n$ such that $L(x) = \sum_{i=1}^{n} a_i x_i$ for all $x \in \mathbb{R}^n$.

4. Let S be an n by n positive definite symmetric (real) matrix with rank m. Prove that there exist two real matrices A and B such that A is n by m, B is m by n and $S = AB$.

5. Suppose A and B are real $n \times n$ matrices. If A is similar to B, then e^A is similar e^B.