Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Tuesday, January 24, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Prove that a continuous function $f : [0, 1] \to \mathbb{R}$ satisfies

$$\int_0^1 f(x)\psi(x)\,dx = \int_0^1 f(x)\,dx \int_0^1 \psi(x)\,dx$$

for all continuous functions $\psi : [0, 1] \to \mathbb{R}$ if and only if f is constant.

2. Let A and B be two $n \times n$ real matrices. Show that if $AB = 0$ then

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq n.$$

3. Let A and B be $n \times m$ real matrices. Prove that a necessary and sufficient condition that there exists an $m \times m$ real matrix C such that $AC = B$ is that the column space of B is a subspace of the column space of A. (The *column space* of a matrix is the vector space spanned by its columns.)

4. Prove that there is a unique solution of the integro-differential equation

$$f'(x) = f(x) + \int_0^1 f(y)\,dy, \quad x \in \mathbb{R},$$

with f twice-differentiable and $f(0) = 1$, and find that solution.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies $f'(x) \leq \frac{1}{2}$ for all x.

   Prove that f has a unique fixed point, that is, that there exists one and only one real value a such that $f(a) = a$.

6. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$, and then compute the value of $\sum_{n=1}^{\infty} \frac{1}{n3^n}$.

7. Show that if E and F are positive definite, then

$$\det(E+F) \geq \det E + \det F.$$

You may use without proof the existence of a positive definite square root $E^{1/2}$ of a positive definite matrix E.

8. Suppose X and Y are jointly continuous random variables having joint probability density function (pdf)

$$f(x,y) = \begin{cases} x+y, & \text{if } 0 < x < 1 \text{ and } 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find the pdf f_W of $W := X + Y$.

9. Let $f_n:[0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of continuous functions and let $f:[0,1] \to \mathbb{R}$ be another continuous function. Show that f_n converges uniformly to f if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ such that x_n converges to some $x \in [0,1]$, we have $\lim_{n\to\infty} f_n(x_n) = f(x)$.

10. Take $N > 1$ to be a positive integer and $x_1, \ldots, x_N$ to be real numbers. Let $\mu := \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\sigma^2 := \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$. Suppose we draw I_1 uniformly from $\{1, \ldots, N\}$ and then, conditionally given $I_1 = i$, we draw I_2 uniformly from $\{1, \ldots, N\}\setminus\{i\}$. Define $X_i := x_{I_i}$ for $i = 1, 2$. Show that $\mathrm{Cov}(X_1, X_2) = -\sigma^2/(N-1)$.

11. Let X denote the number of different days of the year that are birthdays of four persons selected randomly. Calculate $\mathbb{E}\,X$. *You may assume that persons' birthdays are independent and uniformly distributed over 365 days of the year.*

12. Prove that there do not exist a number $\epsilon > 0$ and a *real* matrix A that satisfy

$$A^{100} = \begin{bmatrix} -1 & 0 \\ 0 & -(1+\epsilon) \end{bmatrix}.$$

13. The ages of prospective married parents at a certain hospital can be approximated by a bivariate normal distribution with parameters $\mu_X = 28.2$, $\sigma_X = 6.0$, $\mu_Y = 31.5$, $\sigma_Y = 7.0$, and $\rho = 0.80$. (The parameters having label X refer to the pregnant women and those with label Y to the prospective father. The quantities μ are means and the quantities σ are standard deviations; ρ is the correlation.) For this hospital:

    (a) Consider the proportion of pregnant women who are over 30. Is this proportion closest to 10%, 40%, 60%, or 90%?

    (b) Consider the proportion of prospective fathers aged 35 who have wives over 30. Is this proportion closest to 10%, 40%, 60%, or 90%?

    HINT: Recall—no proof need be given—that if (X, Y) has a bivariate normal distribution with means (μ_X, μ_Y), variances (σ_X^2, σ_Y^2), and correlation ρ, then the conditional distribution of X given $Y = y$ is normal with

    $$\text{mean} \quad \mu_{X|Y=y} = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y) \quad \text{and} \quad \text{variance} \quad \sigma^2_{X|Y=y} = \sigma^2_X(1 - \rho^2).$$

    You should not require a table of the normal distribution to solve this problem.

14. A vector x in $\mathbb{R}^n$ has length 6. A vector y in $\mathbb{R}^n$ has the property that for every pair of scalars a and b the vectors $ax + by$ and $4bx - 9ay$ are orthogonal. Compute the length of y and of $2x + 3y$.

15. Let X_1 and X_2 be independent normal random variables, each with mean zero but (perhaps) different variances ν_1 and ν_2. Correspondingly, we write the probability density function for X_j as $\varphi_{X_j}(\cdot) = \varphi(\cdot; 0, \nu_j)$. Derive the maximally-simplified closed-form expression for

    $$\mathbb{E}[X_1 | X_1 + X_2 = s]$$

    in terms of just s, ν_1, and ν_2.