Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Wednesday, January 20, 2016

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Evaluate:

$$\lim_{n\to\infty}\left(\frac{1}{\log(1+\frac{1}{n})}-n\right)$$

where log is the natural (base-e) logarithm.

Solution: Note that for x small

$$\log(1+x)=x-\frac{1}{2}x^2+O(x^3).$$

We then calculate:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{\log(1+\frac{1}{n})}-n\right)&=\lim_{n\to\infty}\frac{1-n\log(1+\frac{1}{n})}{\log(1+\frac{1}{n})}\\&=\lim_{n\to\infty}\frac{1-n\left[\frac{1}{n}-\frac{1}{2n^2}+O(n^{-3})\right]}{\frac{1}{n}+O(n^{-2})}\\&=\lim_{n\to\infty}\frac{\frac{1}{2n}+O(n^{-2})}{\frac{1}{n}+O(n^{-2})}=\frac{1}{2}.\end{aligned}$$

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2. Let V be a vector space over $\mathbb{R}$ and suppose $L:V\to\mathbb{R}$ is a linear map and $v\in V$ is not in the nullspace of L, then for every $w\in V$ there exists a unique decomposition $w=cv+u$ where u is in the nullspace of L and $c\in\mathbb{R}$.

Solution: Define $u=w-\frac{L(w)}{L(v)}v$ and $c=\frac{L(w)}{L(v)}$, then

$$cv+u=\frac{L(w)}{L(v)}v+w-\frac{L(w)}{L(v)}v=w,$$

and

$$L(u)=L(w-cv)=L(w)-cL(v)=L(w)-\frac{L(w)}{L(v)}L(v)=0,$$

that is, u lies in the nullspace of L.

For uniqueness, if $w\in V$ and $w=cv+u$ where $L(u)=0$ then linearity gives

$$L(w)=cL(v)+L(u)=cL(v)$$

so $c=L(w)/L(v)$ and $u=w-cv$ so we conclude that c and u are uniquely determined.

3. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables with a continuous distribution function. Let N be the time at which the sequence stops decreasing, that is, let $N \geq 2$ be such that $X_1 \geq X_2 \geq X_3 \geq \cdots \geq X_{N-1}$ and $X_{N-1} < X_N$. Find the value of $E[N]$.

   Hint: First find $P[N \geq n]$.

   Solution: Since the distribution is continuous, the probability that any two of the random variables are equal is zero.

   By symmetry, all possible orderings of $X_1, X_2, X_3, \ldots, X_n$ are equally likely.

   Combining these, we see that

$$P[N \geq n] = P\left[X_1 \geq X_2 \geq \cdots \geq X_{n-1}\right] = \frac{1}{(n-1)!}.$$

   Using the tail probability sum representation of the mean, we have

$$E[N] = \sum_{n=1}^{\infty} P[N \geq n] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e,$$

   where the last equality follows directly from the Taylor series for the exponential function e^x evaluated at $x = 1$.

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4. Let A and B be $n \times n$ real matrices. Suppose that the columns of A form an orthonormal basis for $\mathbb{R}^n$ and, likewise, the columns of B form a (possibly different) orthonormal basis for $\mathbb{R}^n$.

   Prove that the columns of AB are also an orthonormal basis for $\mathbb{R}^n$.

   Solution: That the columns of A are an orthonormal basis is equivalent to the statement that $A^t A = I$; that is, $A^{-1} = A^t$. Likewise $B^{-1} = B^t$.

   We need to show that $(AB)^t(AB) = I$. Here goes:

$$(AB)^t(AB) = B^t A^t AB = B^{-1}A^{-1}AB = B^{-1}B = I$$

   as required. In other words, the product of orthogonal matrices is orthogonal.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(y) - f(x)| \le |y - x|^2 \text{ for all } x, y \in \mathbb{R}.$$

Show that f is constant.

Solution: For any pair of distinct points $x, y \in \mathbb{R}$ assume without loss of generality that $x < y$. For any positive integer n, we can write

$$f(y) - f(x) = \sum_{i=1}^{n} f(x + i(y-x)/n) - f(x + (i-1)(y-x)/n)$$

and using the triangle inequality repeatedly gives

$$|f(x) - f(y)| \le \sum_{i=1}^{n} |f(x + i(y-x)/n) - f(x + (i-1)(y-x)/n)| \le n((y-x)/n)^2.$$

Letting $n \to \infty$ we see that $|f(y) - f(x)| = 0$ so $f(y) = f(x)$.

Alternate solution. Fix $y \in \mathbb{R}$. The stated condition is $\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|$ for any $x \ne y$. Therefore, $0 \le \liminf_{x\to y} \left|\frac{f(x)-f(y)}{x-y}\right| \le \limsup_{x\to y} \left|\frac{f(x)-f(y)}{x-y}\right| \le 0$, which implies $\lim_{x\to y} \left|\frac{f(x)-f(y)}{x-y}\right| = |f'(y)| = 0$ or simply, $f'(y) = 0$. But, $f'(y) = 0$ for all y implies $f(y) = c$ for some constant c.

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6. One says that a two-dimensional random vector Z has a uniform distribution on the unit circle

$$S^1 = \{z : \|z\| = 1\},$$

if Z takes values in S^1 and if

$$P(Z \in \text{arc}(z_1, z_2)) = \text{length}(\text{arc}(z_1, z_2))/2\pi,$$

where, for $z_1, z_2 \in S^1$, $\text{arc}(z_1, z_2)$ is the counter-clockwise sub-arc of S^1 between z_1 and z_2.

Prove that, if X and Y are two independent standard Gaussian variables, then Z defined by $Z = (X, Y)/\sqrt{X^2 + Y^2}$ if $(X, Y) \ne (0, 0)$ and $Z = (0, 0)$ otherwise has a uniform distribution on S^1.

Solution: Consider the mapping $\phi : (x, y) \mapsto (r, \theta)$ where r and θ are such that $x = r\cos\theta, y = r\sin\theta$, $r \geq 0$ and $\theta \in [0, 2\pi)$ (with $\phi(0,0) = (0,0)$).

Let $z_1, z_2 \in S^1$, so that $\phi(z_i) = (1, \theta_i)$, $i = 1, 2$. We assume that $0 \leq \theta_1 < \theta_2 < 2\pi$ (all other cases can be deduced from this one by taking the complementary arc, or switching z_1 and z_2). Then

$$P(Z \in \mathrm{arc}(z_1, z_2)) = P((X, Y) \in \Gamma(\theta_1, \theta_2))$$

with $\Gamma(\theta_1, \theta_2)$ the wedge sector in the plane defined by

$$\Gamma(\theta_1, \theta_2) = \{(x, y) : \phi(x, y) = (r, \theta) : \theta_1 \leq \theta \leq \theta_2\},$$

and

$$\begin{aligned} P((X,Y) \in \Gamma(\theta_1, \theta_2)) &= \frac{1}{2\pi} \int_{\Gamma(\theta_1,\theta_2)} e^{-(x^2+y^2)/2} dx dy \\ &= \frac{1}{2\pi} \int_0^{+\infty} \int_{\theta_1}^{\theta_2} e^{-r^2/2} r dr d\theta \\ &= (\theta_2 - \theta_1)/2\pi. \end{aligned}$$

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7. Let $a_k > 0$ be such that $\sum_{k=0}^{\infty} a_k < \infty$. Let $S_k = \sum_{l \geq k} a_l$, and assume that ρ is a C^1 function defined on $[0, +\infty)$ such that $\rho(0) = 0$, where ρ' is positive and decreasing. Prove that
$$\sum_{k=0}^{\infty} \rho'(S_k) a_k < \infty.$$

Solution: Since $\rho' > 0$ and decreasing, we have, for $k \geq 1$

$$\rho(S_{k-1}) - \rho(S_k) = a_k \rho'(S_k^*) \geq a_k \rho'(S_k)$$

with $S_k^* \in [S_k, S_{k-1}]$. This implies

$$\sum_{k=0}^{\infty} a_k \rho'(S_k) \leq a_0 \rho'(S_0) + \sum_{k=1}^{\infty} (\rho(S_k) - \rho(S_{k-1})) = a_0 \rho'(S_0) - \rho(S_0) < \infty$$

since S_k, hence $\rho(S_k)$, tends to 0 when $k \to \infty$.

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8. Suppose A is an $n \times n$ matrix whose entries are independent and identically distributed random variables with

$$P[A_{ij} = 1] = P[A_{ij} = -1] = 1/2, \text{ for all } i, j.$$

Find a simple formula for $E[tr(A^4)]$.

Solution: Since

$$(A^4)_{ij} = \sum_{p,q,r=1}^{n} A_{ip}A_{pq}A_{qr}A_{rj}$$

we have

$$tr(A^4) = \sum_{i,p,q,r=1}^{n} A_{ip}A_{pq}A_{qr}A_{ri}$$

so that by linearity of expectation

$$E[tr(A^4)] = \sum_{i,p,q,r=1}^{n} E\left[A_{ip}A_{pq}A_{qr}A_{ri}\right]$$

By independence and the fact that the matrix entries have zero expectation, we have $E\left[A_{ip}A_{pq}A_{qr}A_{ri}\right] = 0$ unless the number of times each term appears in the product $A_{ip}A_{pq}A_{qr}A_{ri}$ is even. So, either there is one term repeated 4 times, which means that $p = q = r = i$ or there are two terms each appearing twice, which can only happen if $q = i \neq p = r$. Thus

$$E[tr(A^4)] = \sum_{i=1}^{n} E\left[A_{ii}^4\right] + \sum_{i \neq j}^{n} E\left[A_{ij}^2 A_{ji}^2\right] = n + n(n-1) = n^2.$$

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9. Let n be a large real number. We wish to approximate $\sqrt{n+1} - \sqrt{n}$ by a function of the form αn^{β} where α, β are specific numbers (constants). What constants α, β give the "best" approximation in the sense that

$$\sqrt{n+1} - \sqrt{n} \sim \alpha n^{\beta} \quad \text{as } n \to \infty?$$

Here, $a_n \sim b_n$ as $n \to \infty$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Solution: Observe:

$$\begin{aligned}
\sqrt{n+1}-\sqrt{n} &= \sqrt{n}\sqrt{1+\frac{1}{n}}-\sqrt{n} && (1)\\
&= \sqrt{n}\left[\sqrt{1+\frac{1}{n}}-1\right] && (2)\\
&= \frac{\sqrt{n}}{n}\left[\frac{\sqrt{1+\frac{1}{n}}-1}{\frac{1}{n}}\right] && (3)\\
&\sim \frac{1}{\sqrt{n}}\left[\frac{1}{2}\right] && (4)\\
&= \frac{1}{2\sqrt{n}} = \frac{1}{2}n^{-1/2}. && (5)
\end{aligned}$$

Note the step from line (3) to (4) is uses the fact that the derivative of $\sqrt{x}$, evaluated at $x = 1$, is $\frac{1}{2}$ or, alternatively, the binomial expansion of $\sqrt{1+x}$ at $x = 0$.

Thus $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$.

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10. In a sample of radioactive material composed of unstable atoms with exponential decay rate $\lambda > 0$, the number of atoms $N(t)$ to have decayed by time t has a distribution given by a Poisson random variable with parameter λt. Show as a consequence that the total time T_n until the decay of the nth atom has a Gamma distribution for each nonnegative integer n.

Solution: Clearly, by the definitions of $N(t)$ and T_n,

$$P(T_n \leq t) = P(N(t) \geq n).$$

Thus, the cumulative distribution function of T_n is given by

$$F_{T_n}(t) = P(T_n \leq t) = e^{-\lambda t}\sum_{k=n}^{\infty}\frac{(\lambda t)^k}{k!}.$$

Differentiating with respect to t to find the density,

$$\begin{aligned}
p_{T_n}(t) &= -\lambda e^{-\lambda t}\sum_{k=n}^{\infty}\frac{(\lambda t)^k}{k!} + e^{-\lambda t}\sum_{k=n}^{\infty}\frac{\lambda^k t^{k-1}}{(k-1)!}\\
&= \frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t}
\end{aligned}$$

since all but the single term above cancels between the two infinite sums. However, this is precisely the density of a Gamma distribution.

11. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$.

Solution: The given equation can be rewritten as $I = I - A - B + AB$. We can factor the right-hand side as $(I - A)(I - B) = I$. So $I - A$ and $I - B$ are inverses. Hence also $(I - B)(I - A) = I$. Expanding, we get $A + B = BA$. Therefore

$$AB = A + B = BA.$$

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12. In a town of $n+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, and so on. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without being repeated to any person.

Solution:

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-(r-1)}{n} = \frac{(n)_r}{n^r}.$$

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13. Let X and Y be independent random variables each having the exponential distribution with parameter $\lambda = 1$. Let $Z = \min\{X, Y\}/\max\{X, Y\}$. Find the pdf of Z.

Solution: We first compute the cdf of Z: for $0 < z < 1$,

$$\begin{aligned}
F_Z(z) = P(Z \le z) &= P(\min\{X,Y\} \le z \max\{X,Y\}) \\
&= 1 - P(\min\{X,Y\} > z \max\{X,Y\}) \\
&= 1 - P(X > z\max\{X,Y\}, Y > z\max\{X,Y\}) \\
&= 1 - P(X > zX, X > zY, Y > zX, Y > zY) \\
&= 1 - P(X > zY, Y > zX) \\
&= 1 - P(zY < X < Y/z) = 1 - E(P(zY < X < Y/z|Y)) \\
&= 1 - E\left(\int_{zY}^{Y/z} e^{-x}\,dx\right) = 1 - E\left(e^{-zY} - e^{-Y/z}\right) \\
&= 1 - \int_0^\infty e^{-zy}e^{-y}\,dy + \int_0^\infty e^{-y/z}e^{-y}\,dy \\
&= 1 - \frac{1}{z+1} + \frac{1}{\frac{1}{z}+1} = \frac{2z}{z+1}.
\end{aligned}$$

Since $\min\{X,Y\} \le \max\{X,Y\}$, we have $F_Z(z) = 1$ for $z \ge 1$. We also clearly have $F_Z(z) = 0$ for $z \le 0$ since X and Y are positive random variables. Finally, $f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{2}{(z+1)^2}$ for $0 < z < 1$, and $f_Z(z) = 0$ otherwise.

Alternate solution. The joint pdf of $U = \min\{X,Y\}$ and $V = \max\{X,Y\}$ is $f_{U,V}(u,v) = 2!e^{-(u+v)}$ for $0 < u < v < \infty$. Consider the transformation $z = u/v$ and $w = v$, which has the inverse transformation $u = zw$ and $v = w$ having Jacobian $J(z,w) = w$. Since $w > 0$, $|J| = w$. Therefore, the joint pdf of Z and W is $f_{Z,W}(z,w) = f_{U,V}(zw,w)w = 2we^{-(z+1)w}$ for $0 < z < 1$ and $w > 0$. Finally,

$$f_Z(z) = \int_0^\infty 2we^{-(z+1)w}\,dw = 2(z+1)^{-2}, \text{ for } 0 < z < 1.$$

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14. Suppose (X,d) is a compact metric space and let $\varepsilon > 0$. Show that there is a finite subset $\{x_1, x_2, \ldots, x_N\} \subset X$ such that for every $y \in X$, $d(y, x_i) < \varepsilon$ for some $i = 1, 2, \ldots, N$.

Solution: Let $B(x,\varepsilon) = \{z \in X : d(z,x) < \varepsilon\}$ be the open ball of radius ε centered at $x \in X$. Since $X \subset \bigcup_{x\in X} B(x,\varepsilon)$, we see that the set of open balls centered at each $x \in X$ forms an open cover of X. Consequently, the open cover can be reduced to a finite subcover, that is, there is a finite number of these open balls $B(x_1,\varepsilon), B(x_2,\varepsilon), \ldots, B(x_N,\varepsilon)$ whose union still contains X. Therefore, every $y \in X$ belongs to at least one of these balls, that is, $d(y,x_i) < \varepsilon$ for at least one x_i.

15. Let A be a real $n \times n$ matrix. If λ_1 and λ_2 are *distinct* eigenvalues with corresponding eigenvectors x_1 and x_2, prove that $x := x_1 + x_2$ cannot be an eigenvector of A.

Solution: From the assumptions of the question, we have

$$Ax_1 = \lambda_1 x_1 \text{ and } Ax_2 = \lambda_2 x_2. \tag{6}$$

Now, for a proof by contradiction, suppose that $x = x_1 + x_2$ is an eigenvector of A; let the eigenvalue be λ. Combining this with (6) means that

$$\lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2) = \lambda x = Ax = A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda_1 x_1 + \lambda_2 x_2.$$

After moving all terms to the left side, we have

$$(\lambda - \lambda_1)x_1 + (\lambda - \lambda_2)x_2 = 0.$$

Since x_1 and x_2 are linearly independent (eigenvectors corresponding to distinct eigenvalues are linearly independent), we must conclude that $\lambda = \lambda_1 = \lambda_2$. This contradicts the fact that λ_1 and λ_2 are distinct, which completes the proof.