Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Wednesday, January 20, 2016

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Evaluate:
$$\lim_{n \to \infty} \left(\frac{1}{\log(1 + \frac{1}{n})} - n \right)$$
   where log is the natural (base-e) logarithm.

2. Let V be a vector space over $\mathbb{R}$ and suppose $L : V \to \mathbb{R}$ is a linear map and $v \in V$ is not in the nullspace of L, then for every $w \in V$ there exists a unique decomposition $w = cv + u$ where u is in the nullspace of L and $c \in \mathbb{R}$.

3. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables with a continuous distribution function. Let N be the time at which the sequence stops decreasing, that is, let $N \geq 2$ be such that $X_1 \geq X_2 \geq X_3 \geq \cdots \geq X_{N-1}$ and $X_{N-1} < X_N$. Find the value of $E[N]$.

   Hint: First find $P[N \geq n]$.

4. Let A and B be $n \times n$ real matrices. Suppose that the columns of A form an orthonormal basis for $\mathbb{R}^n$ and, likewise, the columns of B form a (possibly different) orthonormal basis for $\mathbb{R}^n$.

   Prove that the columns of AB are also an orthonormal basis for $\mathbb{R}^n$.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$|f(y) - f(x)| \leq |y - x|^2 \text{ for all } x, y \in \mathbb{R}.$$
   Show that f is constant.

6. One says that a two-dimensional random vector Z has a uniform distribution on the unit circle
$$S^1 = \{z : \|z\| = 1\},$$
   if Z takes values in S^1 and if
$$P(Z \in \text{arc}(z_1, z_2)) = \text{length}(\text{arc}(z_1, z_2))/2\pi,$$

where, for $z_1, z_2 \in S^1$, $\text{arc}(z_1, z_2)$ is the counter-clockwise sub-arc of S^1 between z_1 and z_2.

Prove that, if X and Y are two independent standard Gaussian variables, then Z defined by $Z = (X, Y)/\sqrt{X^2 + Y^2}$ if $(X, Y) \neq (0, 0)$ and $Z = (0, 0)$ otherwise has a uniform distribution on S^1.

7. Let $a_k > 0$ be such that $\sum_{k=0}^{\infty} a_k < \infty$. Let $S_k = \sum_{l \geq k} a_l$, and assume that ρ is a C^1 function defined on $[0, +\infty)$ such that $\rho(0) = 0$, where ρ' is positive and decreasing. Prove that

$$\sum_{k=0}^{\infty} \rho'(S_k) a_k < \infty.$$

8. Suppose A is an $n \times n$ matrix whose entries are independent and identically distributed random variables with

$$P[A_{ij} = 1] = P[A_{ij} = -1] = 1/2, \text{ for all } i, j.$$

Find a simple formula for $E[tr(A^4)]$.

9. Let n be a large real number. We wish to approximate $\sqrt{n+1} - \sqrt{n}$ by a function of the form αn^β where α, β are specific numbers (constants). What constants α, β give the "best" approximation in the sense that

$$\sqrt{n+1} - \sqrt{n} \sim \alpha n^\beta \quad \text{as } n \to \infty?$$

Here, $a_n \sim b_n$ as $n \to \infty$ means $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$.

10. In a sample of radioactive material composed of unstable atoms with exponential decay rate $\lambda > 0$, the number of atoms $N(t)$ to have decayed by time t has a distribution given by a Poisson random variable with parameter λt. Show as a consequence that the total time T_n until the decay of the nth atom has a Gamma distribution for each nonnegative integer n.

11. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$.

12. In a town of $n+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, and so on. At each step the recipient of the rumor is chosen at random from the n people available. Find the probability that the rumor will be told r times without being repeated to any person.

13. Let X and Y be independent random variables each having the exponential distribution with parameter $\lambda = 1$. Let $Z = \min\{X, Y\}/\max\{X, Y\}$. Find the pdf of Z.

14. Suppose (X, d) is a compact metric space and let $\varepsilon > 0$. Show that there is a finite subset $\{x_1, x_2, \ldots, x_N\} \subset X$ such that for every $y \in X$, $d(y, x_i) < \varepsilon$ for some $i = 1, 2, \ldots, N$.

15. Let A be a real $n \times n$ matrix. If λ_1 and λ_2 are *distinct* eigenvalues with corresponding eigenvectors x_1 and x_2, prove that $x := x_1 + x_2$ cannot be an eigenvector of A.