Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Wednesday, January 21, 2015

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let I be the $k \times k$ identity matrix and J the $k \times k$ matrix of all ones with $k \geq 2$. Define a $k \times k$ real matrix $C = (a-b)I + bJ$ for some real a and b. Show that C has an inverse if and only if $a \neq b$ and $a \neq -(k-1)b$.

Solution: C will be invertible if and only if $\det(C) \neq 0$. We proceed to compute the determinant of C:

$$C = \begin{pmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{pmatrix}.$$

By performing elementary row (and column) operations we will not change the value of the determinant of C. Create a matrix C^* from C by subtracting the second row of C from the first row of C, then subtract the third row of C from the second row of C; etc. The result is the matrix

$$C^* = \begin{pmatrix} a-b & b-a & 0 & \cdots & 0 & 0 \\ 0 & a-b & b-a & \cdots & 0 & 0 \\ 0 & 0 & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a-b & b-a \\ b & b & b & \cdots & b & a \end{pmatrix},$$

which has the same determinant as C. By adding the first column of C^* to the second column of C^*, then adding the second column of this matrix to the third column of C^*, and so on, we arrive at the lower triangular matrix

$$C^{**} = \begin{pmatrix} a-b & 0 & 0 & \cdots & 0 & 0 \\ 0 & a-b & 0 & \cdots & 0 & 0 \\ 0 & 0 & a-b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a-b & 0 \\ b & 2b & 3b & \cdots & (k-1)b & a+(k-1)b \end{pmatrix}$$

whose determinant is $(a+(k-1)b)(a-b)^{k-1}$ which is non-zero exactly when $a \neq b$ and $a \neq -(k-1)b$.

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2. (a) Prove the combinatorial identity

$$\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$$

for $m \geq 1$.

(b) Prove the identity

$$\binom{n+r}{n} = \sum_{j=0}^{n} \binom{j+r-1}{j}$$

for $n \geq 0$ and $r \geq 1$.

Solution:

(a) Here is a combinatorial proof. Given a set of m objects, mark one as "special". The left-hand side is the number of subsets of size k. The first term on the right is the number of such subsets that include the special object, while the second term on the right is the number of such subsets that exclude the special object.

(b) One way to prove the identity is by induction on n using the result of part (a). For $n = 0$ the desired result $\binom{r}{0} = \binom{r-1}{0}$ is clear, since it reduces to $1 = 1$. Now suppose that the desired identity holds for $n - 1 \geq 0$. Then

$$\begin{aligned}
\binom{n+r}{n} &= \binom{n-1+r}{n-1} + \binom{n+r-1}{n} && \text{by part (a)} \\
&= \sum_{j=0}^{n-1} \binom{j+r-1}{j} + \binom{n+r-1}{n} && \text{by induction} \\
&= \sum_{j=0}^{n} \binom{j+r-1}{j}.
\end{aligned}$$

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3. Let X be a real random variable that only takes positive values (i.e., $X > 0$ always). Does the equation

$$\log E[X] = E[\log X] \qquad (*)$$

hold always, sometimes, or never?

That is to say:

- If you answer *always*, then you must prove that $(*)$ holds for all positive random variables X.

- If you answer *sometimes*, then you must give an example where $(*)$ holds and an example where $(*)$ fails.
- And if you answer *never*, then you must prove that $(*)$ fails for all positive random variables X.

Solution: Equation $(*)$ holds sometimes.

Equation $(*)$ holds if X is a constant random variable, say, $X = x$ (with probability one). Then both sides of $(*)$ equal $\log x$.

But if $X = 1$ with probability $\frac{1}{2}$ and e with probability $\frac{1}{2}$, then

$$\log E[X] = \log\left(\frac{1+e}{2}\right) \approx 0.62,$$
$$E[\log X] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = 0.5$$

and so $(*)$ fails.

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4. Let f be a real-valued function defined on $\mathbb{R}$. Prove that if f is continuous, then $f^{-1}(S)$ is an open set for each open set $S \subset \mathbb{R}$.

Solution: Let $S \subset \mathbb{R}$ be an open set. Let $x \in f^{-1}(S)$. Then $f(x) \in S$, and there exists $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset S$. Since f is continuous, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$. Thus, for every $y \in (x-\delta, x+\delta)$, it follows that $f(y) \in (f(x) - \epsilon, f(x) + \epsilon) \subset S$. Therefore $(x - \delta, x + \delta) \subset f^{-1}(S)$, implying that S is open.

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5. Suppose X and Y are random variables with the property that $P(|X - Y| < \varepsilon) = 1$ for some $\varepsilon > 0$. Show that if X has finite expectation, then Y has finite expectation and $|E(X) - E(Y)| \leq \varepsilon$.

Solution: Since $|X - Y|$ is a nonnegative random variable and $P(|X - Y| \geq \varepsilon) = 0$

it follows that

$$\begin{aligned} E(|X-Y|) &= \int_0^\infty P(|X-Y| \geq t)\, dt \\ &= \int_0^\varepsilon P(|X-Y| \geq t)\, dt \\ &\leq \int_0^\varepsilon 1\, dt = \varepsilon. \end{aligned}$$

Further, $|Y| = |X - Y - X| \leq |X - Y| + |X|$ shows that Y has finite expectation. Finally, $|E(X) - E(Y)| = |E(X - Y)| \leq E(|X - Y|) \leq \varepsilon$.

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6. Let A be an $n \times n$ real matrix, show that if $A^2 = A$ then

$$\text{rank}(A) + \text{rank}(I - A) = n.$$

Solution: For any $x \in R^n$,

$$x = Ax + (I - A)x.$$

Therefore we have $R^n = \text{range}(A) + \text{range}(I - A)$ and

$$n = \dim(\text{range}(A) + \text{range}(I - A)) \leq \text{rank}(A) + \text{rank}(I - A).$$

On the other hand, it follows from $A(I - A) = 0$ that $\text{range}(I - A) \subset \text{null}(A)$. Hence,

$$\text{rank}(A) + \text{rank}(I - A) \leq \text{rank}(A) + \dim(\text{null}(A)) = n.$$

Alternate solution: Because $A^2 - A = 0$, the eigenvalues of A are roots of $\lambda^2 - \lambda = 0$. Therefore, both A and $I - A$ have eigenvalues 1 or 0. We have

$$\text{rank}(A) = \text{ multiplicity of eigenvalue 1 of } A$$

$$\text{rank}(I - A) = \text{ multiplicity of eigenvalue 1 of } I - A.$$

Therefore, $\text{rank}(A) + \text{rank}(I - A) = n$.

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7. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be such that there exists $0 < \alpha < 1$ with the property that

$$|f(x) - f(y)| \leq \alpha |x - y| \text{ for all } x, y \in \mathbb{R}^k.$$

Show that there exists a unique $x^* \in \mathbb{R}^k$ such that $f(x^*) = x^*$.

Solution: Let $x_0 \in \mathbb{R}^k$ be any point in $\mathbb{R}^k$. We consider the sequence of points $x_n := f^n(x_0)$, $n = 1, 2, \ldots$, where $f^1(x_0) = f(x_0)$ and for $n > 1$, $f^n(x_0) = f(f^{n-1}(x_0))$. By induction, it can be shown that $|x_{n+1} - x_n| \leq \alpha^n |x_1 - x_0|$. Thus for $1 \leq m < n < \infty$ we have

$$|x_n - x_m| \leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \leq \sum_{k=m}^{n-1} \alpha^k |x_1 - x_0| \leq |x_1 - x_0| \sum_{k=m}^{\infty} \alpha^k = \frac{\alpha^m}{1 - \alpha} |x_1 - x_0|,$$

which doesn't depend on n and vanishes in the limit as $m \to \infty$. This shows that the sequence (x_n) is a Cauchy sequence. Since $\mathbb{R}^k$ is complete, every Cauchy sequence converges, and therefore x_n converges to some $x^* \in \mathbb{R}^k$. We show that $f(x^*) = x^*$. Suppose to the contrary that $f(x^*) = y^* \neq x^*$. Let $r = |x^* - y^*|$ and choose $N \in \mathbb{N}$ such that $|x_n - x^*| < \frac{r}{2}$ for all $n \geq N$. Then $\frac{r}{2} > \alpha \frac{r}{2} > \alpha |x_N - x^*| \geq |f(x_N) - f(x^*)| = |x_{N+1} - y^*|$. Also, $|x_{N+1} - x^*| < \frac{r}{2}$. But then

$$r = |x^* - y^*| \leq |x^* - x_{N+1}| + |x_{N+1} - y^*| < \frac{r}{2} + \frac{r}{2} = r$$

which is a contradiction.

The uniqueness of x^* follows from the following argument. Suppose there exists $y^* \neq x^*$ such that $f(y^*) = y^*$. Then $|f(x^*) - f(y^*)| \leq \alpha |x^* - y^*| < |x^* - y^*| = |f(x^*) - f(y^*)|$, which is a contradiction.

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8. Prove that an $n \times n$ real matrix A, orthogonal ($A^\top = A^{-1}$) and unimodular ($\det(A) = 1$) for n odd leaves invariant at least one vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$.

Solution: Since A is unitary, all of its eigenvalues λ satisfy $|\lambda| = 1$. Since the characteristic polynomial of A is of odd degree, it has at least one real root λ, thus $= 1$ or -1, and an odd number of real eigenvalues. The corresponding eigenvectors $\mathbf{x}$ can be chosen to be real. If one supposes that all of the real eigenvalues of A are -1, then $\det(A) = -1$ since the complex conjugate pairs contribute factors of $|\lambda|^2 = 1$. Thus, A has at least one eigenvalue $+1$ and the corresponding eigenvector satisfies $A\mathbf{x} = \mathbf{x}$.

9. Evaluate:

$$\lim_{n\to\infty}\left[\frac{1}{\log(1+\frac{1}{n})}-n\right]$$

where log is the natural (base-e) logarithm.

Solution: Note that for x small

$$\log(1+x) = x - \frac{1}{2}x^2 + O(x^3).$$

We then calculate:

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{1}{\log(1+\frac{1}{n})}-n\right] &= \lim_{n\to\infty}\frac{1-n\log(1+\frac{1}{n})}{\log(1+\frac{1}{n})}\\ &= \lim_{n\to\infty}\frac{1-n\left[\frac{1}{n}-\frac{1}{2n^2}+O(n^{-3})\right]}{\frac{1}{n}+O(n^{-2})}\\ &= \lim_{n\to\infty}\frac{\frac{1}{2n}+O(n^{-2})}{\frac{1}{n}+O(n^{-2})} = \frac{1}{2}.\end{aligned}$$

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10. Let A be an $n\times n$ real symmetric matrix such that $A^k = A^{k+1}$ for some positive integer k. Show that $A^2 = A$.

Solution: Let P be an orthogonal matrix such that $P^TAP = D$, where D is the diagonal matrix of the eigenvalues of A. Raising both sides of the preceding equation to the kth power gives $(P^TAP)^k = (P^TAP)(P^TAP)\cdots(P^TAP) = P^TA^kP = D^k$. Similarly, $P^TA^{k+1}P = D^{k+1}$, and since $A^k = A^{k+1}$ we must have $D^k = D^{k+1}$. This shows that each element of the diagonal of D is either 0 or 1. Thus, $D^2 = D$, which in turn implies $A^2 = A$.

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11. Compute the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

Solution: The prime factorization of 10^{99} is $2^{99}\cdot 5^{99}$, so all divisors of 10^{99} have the form $2^a\cdot 5^b$ where a and b are integers with $0\le a,b\le 99$. Since there are 100 choices for each of a and b, 10^{99} has 100^2 positive integer divisors. Of these, the multiples of $10^{88} = 2^{88} \cdot 5^{88}$ must satisfy the inequalities $88 \le a, b \le 99$. Thus, there are 12 choices for each of a and b, so 12^2 of the 100^2 divisors of 10^{99} are multiples of 10^{88}. Consequently, the desired probability is

$$\frac{12^2}{100^2} = \frac{9}{625}.$$

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12. Let X and Y be independent standard Gaussian variables. Show that the probability density function of $Z = X/Y$ has the form

$$f(z) = \frac{1}{\pi(1+z^2)}\,, \qquad z \in \mathbb{R}\,.$$

(This distribution is known as the standard Cauchy distribution.)

Solution: Let $Z = \frac{X}{Y}$ and $W = X$ and consider the transformation $(X,Y) \to (Z,W)$, i.e.,

$$Z(X,Y) = \frac{X}{Y}, \quad W(X,Y) = X.$$

The inverse transformation is $X = X(Z,W) = W$, $Y = Y(Z,W) = \frac{W}{Z}$, and its Jacobian is $\frac{w}{z^2}$. Therefore, $f_{Z,W}(z,w) = f_{X,Y}(w, \frac{w}{z})|\frac{w}{z^2}|$ and

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_{Z,W}(w, \frac{w}{z})\frac{w}{z^2}\,dw + \int_{-\infty}^0 f_{Z,W}(w, \frac{w}{z}) \cdot \frac{-w}{z^2}\,dw \\
&= \int_0^\infty \frac{1}{2\pi} e^{-\frac{w^2}{2}} e^{-\frac{w^2}{2z^2}} \frac{w}{z^2}\,dw + \int_{-\infty}^0 \frac{1}{2\pi} e^{-\frac{w^2}{2}} e^{-\frac{w^2}{2z^2}} \cdot \frac{-w}{z^2}\,dw \\
&= 2\int_0^\infty \frac{1}{2\pi} e^{-\frac{w^2(1+\frac{1}{z^2})}{2}} \frac{w}{z^2}\,dw \\
&= \frac{1}{\pi z^2}\int_0^\infty w e^{-\frac{w^2(1+\frac{1}{z^2})}{2}}\,dw = \frac{1}{\pi z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{\pi(z^2+1)}.
\end{aligned}$$

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13. Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that $Ax = b$ has a solution $x \in \mathbb{R}^n$ if and only if b is orthogonal to all $y \in \mathbb{R}^m$ such that $A^T y = 0$.

Solution: We are asked to show that b is in the range of A if and only if b is in the orthogonal complement of the null space of A^T, i.e., that $\text{range}(A) = \text{null}(A^T)^\perp$, or, equivalently, $\text{range}(A)^\perp = \text{null}(A^T)$.

First suppose $y \in \text{null}(A^T)$ (i.e., $A^T y = 0$) and $b \in \text{range}(A)$ (i.e., $Ax = b$ for some $x \in \mathbb{R}^n$). Then $y^T b = y^T Ax = (A^T y)^T x = 0^T x = 0$, and so $y \in \text{range}(A)^\perp$. This shows $\text{null}(A^T) \subseteq \text{range}(A)^\perp$.

On the other hand, for any $y \in \text{range}(A)^\perp$ and $x \in \mathbb{R}^n$ we have $Ax \in \text{range}(A)$ and so $0 = (Ax)^T y = x^T A^T y$. Since this is true for *any* $x \in \mathbb{R}^n$, we must have $A^T y = 0$, i.e., $y \in \text{null}(A^T)$. This shows $\text{range}(A)^\perp \subseteq \text{null}(A^T)$.

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14. Let (a_n) be a real sequence such that $\sum_{n=1}^{\infty} a_n = c \in \mathbb{R}$. Use Cesàro's theorem to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (N+1-k) a_k = c.$$

Solution: Define the partial sums $s_n := \sum_{k=1}^{n} a_k$. Observe first that summation by parts gives

$$\sum_{k=1}^{N} (N+1-k) a_k = \sum_{k=1}^{N} a_k \sum_{n=k}^{N} 1 = \sum_{n=1}^{N} \sum_{k=1}^{n} a_k = \sum_{n=1}^{N} s_n.$$

Since by assumption $s_n \to c$ as $n \to \infty$, it follows from Cesàro's theorem that

$$\frac{1}{N} \sum_{k=1}^{N} (N+1-k) a_k = \frac{1}{N} \sum_{n=1}^{N} s_n \to c$$

as $N \to \infty$.

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15. Show that for any three real numbers a, b, and c, the following inequality holds:

$$(\tfrac{1}{2}\ a + \tfrac{1}{3}\ b + \tfrac{1}{6}\ c)^2 \le \tfrac{1}{2}\ a^2 + \tfrac{1}{3}\ b^2 + \tfrac{1}{6}\ c^2.$$

Solution: We apply the Cauchy–Schwarz inequality to get

$$\begin{aligned}
&(\tfrac{1}{2}a + \tfrac{1}{3}b + \tfrac{1}{6}c)^2 \\
&= (\tfrac{1}{\sqrt{2}}(\tfrac{1}{\sqrt{2}}a) + \tfrac{1}{\sqrt{3}}(\tfrac{1}{\sqrt{3}}b) + \tfrac{1}{\sqrt{6}}(\tfrac{1}{\sqrt{6}}c))^2 \\
&\le ((\tfrac{1}{\sqrt{2}})^2 + (\tfrac{1}{\sqrt{3}})^2 + (\tfrac{1}{\sqrt{6}})^2)((\tfrac{1}{\sqrt{2}}a)^2 + (\tfrac{1}{\sqrt{3}}b)^2 + (\tfrac{1}{\sqrt{6}}c)^2) \\
&= \tfrac{1}{2}a^2 + \tfrac{1}{3}b^2 + \tfrac{1}{6}c^2.
\end{aligned}$$
