Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Wednesday, January 21, 2015

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let I be the $k \times k$ identity matrix and J the $k \times k$ matrix of all ones with $k \geq 2$. Define a $k \times k$ real matrix $C = (a - b)I + bJ$ for some real a and b. Show that C has an inverse if and only if $a \neq b$ and $a \neq -(k-1)b$.

2. (a) Prove the combinatorial identity

$$\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$$

for $m \geq 1$.

(b) Prove the identity

$$\binom{n+r}{n} = \sum_{j=0}^{n} \binom{j+r-1}{j}$$

for $n \geq 0$ and $r \geq 1$.

3. Let X be a real random variable that only takes positive values (i.e., $X > 0$ always). Does the equation

$$\log E[X] = E[\log X] \qquad (*)$$

hold always, sometimes, or never?

That is to say:

- If you answer *always*, then you must prove that $(*)$ holds for all positive random variables X.
- If you answer *sometimes*, then you must give an example where $(*)$ holds and an example where $(*)$ fails.
- And if you answer *never*, then you must prove that $(*)$ fails for all positive random variables X.

4. Let f be a real-valued function defined on $\mathbb{R}$. Prove that if f is continuous, then $f^{-1}(S)$ is an open set for each open set $S \subset \mathbb{R}$.

5. Suppose X and Y are random variables with the property that $P(|X - Y| < \varepsilon) = 1$ for some $\varepsilon > 0$. Show that if X has finite expectation, then Y has finite expectation and $|E(X) - E(Y)| \leq \varepsilon$.

6. Let A be an $n \times n$ real matrix, show that if $A^2 = A$ then

$$\operatorname{rank}(A) + \operatorname{rank}(I - A) = n.$$

7. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be such that there exists $0 < \alpha < 1$ with the property that

$$|f(x) - f(y)| \leq \alpha |x - y| \text{ for all } x, y \in \mathbb{R}^k.$$

Show that there exists a unique $x^* \in \mathbb{R}^k$ such that $f(x^*) = x^*$.

8. Prove that an $n \times n$ real matrix A, orthogonal ($A^\top = A^{-1}$) and unimodular ($\det(A) = 1$) for n odd leaves invariant at least one vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$.

9. Evaluate:

$$\lim_{n \to \infty} \left[\frac{1}{\log(1 + \frac{1}{n})} - n \right]$$

where log is the natural (base-e) logarithm.

10. Let A be an $n \times n$ real symmetric matrix such that $A^k = A^{k+1}$ for some positive integer k. Show that $A^2 = A$.

11. Compute the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

12. Let X and Y be independent standard Gaussian variables. Show that the probability density function of $Z = X/Y$ has the form

$$f(z) = \frac{1}{\pi(1 + z^2)}, \qquad z \in \mathbb{R}.$$

(This distribution is known as the standard Cauchy distribution.)

13. Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that $Ax = b$ has a solution $x \in \mathbb{R}^n$ if and only if b is orthogonal to all $y \in \mathbb{R}^m$ such that $A^T y = 0$.

14. Let (a_n) be a real sequence such that $\sum_{n=1}^{\infty} a_n = c \in \mathbb{R}$. Use Cesàro's theorem to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (N + 1 - k) a_k = c.$$

15. Show that for any three real numbers a, b, and c, the following inequality holds:

$$(\tfrac{1}{2}\, a + \tfrac{1}{3}\, b + \tfrac{1}{6}\, c)^2 \leq \tfrac{1}{2}\, a^2 + \tfrac{1}{3}\, b^2 + \tfrac{1}{6}\, c^2.$$