Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Winter Session

Tuesday, January 21, 2014

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A vector or matrix is referred to as *integral* if all of its entries are integers. Let A be a nonsingular $n \times n$ integral matrix. Show $A^{-1}b$ is integral for all integral column n-vectors b if and only if $\det(A) = 1$ or -1.

2. For any nonnegative integer-valued random variable, define its probability generating function by $\Phi_X(t) = E(t^X)$ for all real t.

   (a) For any $a > 0$, show that for $0 \le t \le 1$

$$P(X \le a) \le \frac{\Phi_X(t)}{t^a}.$$

   (b) Use the above result to show for a Poisson random variable N with parameter λ and for any $a \in [0, \lambda]$ that

$$P(N \le a) \le e^{-\lambda} \left(\frac{e\lambda}{a}\right)^a$$

3. Assuming that temperature varies continuously with location, prove that there are, at any given time, antipodal points on the equator of the earth that have the same temperature.

4. Suppose the probability that a family will have n children is 2^{-n-1} and that each child is equally likely to be male or female, independently of the other children. What is the conditional probability that a family has at least one child given it has no boys?

5. Suppose the probability that the Dow Jones Stock Index increases today is 0.54, that it increases tomorrow is also 0.54, and that it rises on both days is 0.28. Find (with explanation) the probability that it increases on *neither* day.

6. Let $A_m \in \mathbf{R}^{4\times 4}$, $m = 1, 2, \cdots$, defined by

$$A_m = \begin{bmatrix} 1 & 2^m & 3^m & 4^m \\ 3^m & 1 & 4^m & 2^m \\ 2^m & 4^m & 1 & 3^m \\ 4^m & 3^m & 2^m & 1 \end{bmatrix}$$

Determine for which values of m the matrix A_m is invertible and justify your answer.

7. Consider the one-parameter family of second-order homogeneous linear differential equations of the form

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + [\lambda\rho(x) - q(x)]u = 0, \quad a \le x \le b \tag{1}$$

with *endpoint* (or boundary) *conditions* $u(a) = u(b) = 0$, where λ is a parameter, the functions p, ρ, q are continuous on $[a, b]$, and p, ρ are positive on $[a, b]$. Let $u(x)$ and $v(x)$ be solutions of (1) corresponding to distinct parameters λ and μ, respectively, that satisfy the given endpoint conditions. Show that

$$\int_a^b \rho(x)u(x)v(x)dx = 0$$

8. Let (x_n) be a sequence of positive numbers, and denote the average of the first n entries by

$$\bar{x}_n = (x_1 + \cdots + x_n)/n.$$

Let $N = (n_k)$ be a subsequence of $\mathbb{N}$ with $\lim_{k\to\infty} n_{k+1}/n_k = r > 0$. Prove: if the sequence $(\bar{x}_n)$ converges along N to x, then

$$x/r \le \liminf \bar{x}_n \le \limsup \bar{x}_n \le r \cdot x\ .$$

9. If x is a real number, we use $\lfloor x \rfloor$ to denote its floor, that is, the largest integer that is less than or equal to x. Show that $\lim_{n\to\infty} n!e - \lfloor n!e \rfloor = 0$.

10. Suppose A and B are $n \times n$ matrices such that $I - AB$ is invertible with inverse X. Show that $I - BA$ is invertible and find a simple expression for its inverse.

Hint: To arrive at a good guess for the inverse, write down formal power series expansions for $(I - BA)^{-1}$ and $(I - AB)^{-1}$ to get a candidate for the desired inverse. Then prove that the inverse is correct without using formal power series.

11. Let n be a positive integer. We create a random n-digit (decimal) number as follows:

    First we pick a random number k from $\{0, 1, 2, \ldots, n\}$ uniformly; that is, the probability of picking each k is $1/(n+1)$.

    Then we pick uniformly at random an n-digit number in which exactly k of the digits are 1s and the remaining $n-k$ digits are 9s. That is, given k, all such n-digit numbers are equally likely to be picked.

    Call the resulting value X.

    Question: What is the expected value of X? Justify your answer.

    For example, suppose $n = 5$. If $k = 2$ (with probability 1/6) then the ten numbers 99911, 99191, 99119, 91991, 91919, 91199, 19991, 19919, 19199, 11999 are equally likely (with probability $\frac{1}{6} \cdot \frac{1}{10} = \frac{1}{60}$). However, X takes the value 11111 with probability $\frac{1}{6}$. So not all values of X have the same probability.

12. For which real value(s) of x does the following series converge and diverge:

$$\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right).$$

    Justify your answer.

13. Consider the vector space $V = \{a_0 + a_1x + \cdots + a_nx^n : a_i \text{ real}, n \text{ fixed}\}$ of all polynomials of degree less than or equal to n and the *derivative transformation* $D : V \to V$ sending polynomials in V to their derivatives. Determine the following: the matrix of D with respect to the basis $1, x, x^2, \ldots, x^n$ of V; the rank of $D : V \to V$; and the nullspace of D.

14. Cards are drawn one by one, at random, and without replacement from a standard deck of 52 playing cards. What is the probability that the fourth Heart is drawn on the tenth draw? (Do not simplify to a decimal number.)

15. Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric and positive semidefinite matrices. Show that

$$tr(AB) \geq 0,$$

    where tr denotes the trace, i.e., the sum of the entries on the diagonal.