Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session

Wednesday, August 19, 2015

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Consider families of n children. Let A be the event that a family has children of both sexes, and let B be the event that there is at most one girl in the family. Show that the only value of n for which the events A and B are independent is $n = 3$, assuming that each child has probability $1/2$ of being a boy.

2. Suppose $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are two power series having the same radius of convergence $\rho > 0$ and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for $|x| < \rho$. Prove that the two series are identical, that is, $a_n = b_n$ for $n = 0, 1, 2, \ldots$.

3. A four-digit number is selected at random. What is the probability that its leading digit is strictly larger than its second digit, its second digit is strictly larger than its third digit, and its third digit is strictly larger than its fourth digit. [Note that the leading digit of an n-digit number is nonzero.]

4. Suppose that $f : [-1, 1] \to \mathbb{R}$ is a Riemann integrable function with

$$\int_{-1}^{1} f(x)^2 dx < \infty$$

and that

$$\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} f(x)(x^n + x^{n+1})\ dx$$

for all $n = 0, 1, 2, 3, ...$

Prove that $\int_{-1}^{1} f(x)g(x)dx = 0$ for all functions g continuous on $[-1, 1]$.

5. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive semidefinite matrix for some $n \geq 1$. Prove that $x^T Ax = 0$ if and only if $Ax = 0$.

6. Let $X \in \mathbb{R}^{D \times N}$ for some positive integers D and N, and $Z \in \mathbb{R}^{D \times N}$ be a random matrix whose ND entries are independent, each z_{ij} being a Bernoulli random variable with parameter p, i.e.,

$$P[z_{ij} = k] = (1 - p)^{1-k} p^k \quad \text{for } k \in \{0, 1\},$$

and for all $i = 1, \dots, D$ and $j = 1, \dots, N$. Prove that the matrix

$$\Gamma := Y^T Y + (p-1)\mathrm{diag}(Y^T Y) \quad \text{with} \quad Y := \frac{1}{p}(Z \odot X)$$

satisfies $\mathbb{E}[\Gamma] = X^T X$, where $\odot$ denotes the entry-wise product of matrices and

$$(\mathrm{diag}(Y^T Y))_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ (Y^T Y)_{ij} & \text{if } i = j. \end{cases}$$

7. Let A be a real $n \times n$ positive definite matrix. Prove that $\det(A) \leq (\mathrm{trace}(A)/n)^n$.

   Hint: Use the arithmetic-geometric mean inequality.

8. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(0,0) := 0$ and

$$f(x,y) := \frac{xy(x^2 - y^2)}{x^2 + y^2} \text{ if } (x,y) \neq (0,0).$$

   Prove that $\partial_1 \partial_2 f(0,0)$ and $\partial_2 \partial_1 f(0,0)$ exist, and that they are not equal, where ∂_1 is the partial derivative with respect to the first variable (x) and ∂_2 with respect to the second one (y).

9. (All matrices in this problem are assumed to be real.) Consider symmetric $n \times n$ matrices A and B, and assume that $B - A$ is positive definite. Find necessary and sufficient conditions on $n \times n$ matrices C such that $CBC^T - CAC^T$ is also positive definite.

10. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function that satisfies the following conditions:

    (a) f is continuous,

    (b) f satisfies $f(xy) = f(x) + f(y)$ for all $x, y > 0$,

    (c) $f(1) = 0$, and

    (d) $f(e) = 1$.

Prove that $f(x) = \ln x$.

Hint: Begin by considering x values that are integer powers of e, i.e., $x = e^a$ for $a \in \mathbb{Z}$ and then rational powers of e, i.e., $x = e^{a/b}$ for $a, b \in \mathbb{Z}$, $b \neq 0$.

11. Let $\alpha = 1 + \sqrt{2}$. Because $\alpha > 1$, we know that α^n diverges as $n \to \infty$. However, if we look at the values produced, it is interesting to note that α^n gets closer and closer to being an integer. For example,

$$\left(1 + \sqrt{2}\right)^{20} = 45239073.999999977895215\ldots$$

Explain why this is so, that is, prove that there exists a sequence z_n of integers such that

$$\lim_{n\to\infty} [\alpha^n - z_n] = 0.$$

Hint: Find β such that $\alpha^n + \beta^n$ is an integer for all n.

12. Let $\ldots, Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2, \ldots$ be a (doubly) infinite sequence of independent identically distributed standard normal random variables. For each integer n, let $Y_n = Z_n Z_{n-1}$. Show that the sequence (Y_n) is uncorrelated and *not* pairwise independent. That is, show for all integers $i \neq j$, $E(Y_i Y_j) = E(Y_i)E(Y_j)$, but that there is some integer $i \neq j$ such that Y_i and Y_j are not independent.

13. Let F and G be two subspaces of $\mathbb{R}^n$ such that $\dim(F) + \dim(G) = n$. Prove that there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $\text{Null}(A) = F$ and $\text{Range}(A) = G$.

14. Let S be an n-by-n (real) matrix with rank m. Prove that there exist two real matrices A and B such that A is n-by-m, B is m-by-n, and $S = AB$.

15. A football team consists of 20 offensive and 20 defensive players. The players are to be grouped in groups of 2 for the purpose of determining roommates. If the pairing is done (uniformly) at random, what is the probability that there are no offensive–defensive roommate pairs? Express your answer as a ratio of products of factorials.