Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session

Wednesday, August 28, 2013

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let x, y, z be elements of $\mathbb{R}^n$. Prove that the Euclidean norm $\|\cdot\|$ satisfies

$$\big|\|x-z\| - \|y-z\|\big| \leq \|x-y\|.$$

Solution: Since $x-z = (x-y)+(y-z)$ and $y-z = (y-x)+(x-z)$, subadditivity of the norm gives

$$\|x-z\| \leq \|x-y\| + \|y-z\| \text{ and } \|y-z\| \leq \|y-x\| + \|x-z\| = \|x-y\| + \|x-z\|, \quad (1)$$

or equivalently

$$\|x-z\| - \|y-z\| \leq \|x-y\| \quad \text{and} \quad \|y-z\| - \|x-z\| \leq \|x-y\|, \qquad (2)$$

or equivalently

$$\big|\|x-z\| - \|y-z\|\big| = \max\big\{\|x-z\| - \|y-z\|,\ \|y-z\| - \|x-z\|\big\} \leq \|x-y\|,$$

as desired.

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2. Assume that A and B respectively are m by n and n by m real matrices. Prove that the non-vanishing eigenvalues of AB coincide with the non-vanishing eigenvalues of BA with equal multiplicity.

Solution: If $\lambda \neq 0$ is an eigenvalue of AB, then, for some non-vanishing $u \in \mathbb{R}^m$,

$$ABu = \lambda u, \text{ which implies } BA(Bu) = \lambda(Bu).$$

One must have $Bu \neq 0$ (otherwise, $\lambda = 0$), so λ is also an eigenvalue of BA. This implies (together with the symmetrical argument, inverting the roles of A and B) that the eigenvalues coincides.

In fact, the argument above implies that the restriction of B to the characteristic space of λ for AB, $Null(AB - \lambda I_n)$, is a one-to-one transformation with values in $Null(BA - \lambda I_m)$. This implies that

$$\dim Null(AB - \lambda I_n) \leq \dim Null(BA - \lambda I_m)$$

and the equality holds by symmetry.

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3. Let X and Y be independent random variables, each following a Laplace distribution with probability density function

$$f(t) = \frac{1}{2}e^{-|t|}.$$

Prove that, if $aX + bY$ and $cX + dY$ have the same distribution (with $a, b, c, d \in \mathbb{R}$), then $(a^2, b^2) = (c^2, d^2)$ or $(a^2, b^2) = (d^2, c^2)$.

Solution: Letting

$$m_k = \int_0^{+\infty} t^k e^{-t} dt,$$

one has $m_k = k!$ (use the definition of the Gamma function or integrations by parts to prove this). So, if X follows a Laplace distribution, we get, by symmetry, $E(X) = E(X^3) = 0$ and $E(X^2) = 2$, $E(X^4) = 24$. If X and Y have this distribution, and $Z = aX + bY$, then then $E(Z^2) = 2(a^2 + b^2)$ and

$$E(Z^4) = 24(a^4 + b^4) + 12a^2b^2 = 24(a^2 + b^2)^2 - 12a^2b^2.$$

So, if $aX + bY$ and $cX + dY$ have the same distribution, then

$$a^2 + b^2 = c^2 + d^2$$

and

$$a^2b^2 = c^2d^2.$$

Combining these questions gives

$$(a^2 - c^2)(a^2 - d^2) = a^4 - (c^2 + d^2)a^2 + c^2d^2 = a^4 - (a^2 + b^2)a^2 + a^2b^2 = 0.$$

and similarly

$$(b^2 - c^2)(b^2 - d^2) = b^4 - (c^2 + d^2)b^2 + c^2d^2 = b^4 - (a^2 + b^2)b^2 + a^2b^2 = 0.$$

so $a^2 = c^2$ or $a^2 = d^2$ and $b^2 = c^2$ or $b^2 = d^2$ and the conclusion follows.

Alternatively, for a random variable with the given density function it is straightforward to calculate the characteristic function to be

$$\frac{1}{1 + t^2}.$$

It follows that $aX + bY$ has characteristic function

$$\frac{1}{1 + a^2t^2} \times \frac{1}{1 + b^2t^2}.$$

If $aX + bY$ and $cX + dY$ have the same distribution then their ch. f's are equal and we obtain

$$\frac{1}{1+a^2t^2} \times \frac{1}{1+b^2t^2} = \frac{1}{1+c^2t^2} \times \frac{1}{1+d^2t^2},$$

for all t. Taking the reciprocal and equating polynomials gives $a^2 + b^2 = c^2 + d^2$ and $a^2b^2 = c^2d^2$.

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4. Let $A = [a_{ij}]$ be an $n \times n$ matrix whose diagonal entries are all nonzero, and suppose that $a_{ij} = 0$ if $j \notin \{i, i+1\}$. Find and prove an explicit formula for the entries b_{ij} of the inverse matrix $B = [b_{ij}]$.

Solution: Clearly B exists and is, like A, upper triangular. From successive calculations of $b_{11}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33}, \ldots$, the pattern

$$b_{ij} = \frac{(-1)^{j-i}}{a_{jj}} \prod_{k=i}^{j-1} \frac{a_{k,k+1}}{a_{kk}}, \quad i \le j,$$

emerges. It remains to prove that this formula indeed gives $AB = I$. Indeed, $C := AB$ is upper triangular, and for $i \le j$ we have

$$c_{ij} = \sum_{k:\, i \le k \le j} a_{ik}b_{kj} = a_{ii}b_{ij} + \mathbf{1}(i+1 \le j)a_{i,i+1}b_{i+1,j}.$$

If $j = i$ we find

$$c_{ii} = a_{ii}b_{ii} = a_{ii} \times \frac{1}{a_{ii}} = 1,$$

and if $j \ge i+1$ we find

$$c_{ij} = a_{ii}b_{ij} + a_{i,i+1}b_{i+1,j} = a_{ii}b_{ij} - a_{i,i+1}b_{ij}\frac{a_{ii}}{a_{i,i+1}} = a_{ii}b_{ij} - a_{ii}b_{ij} = 0.$$

Thus $C = I$, completing the proof.

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5. Suppose $f : [0, \infty) \to \mathbb{R}$ is a positive and continuously differentiable function that satisfies the condition:

$$f(x) = 1 + \int_0^x \sqrt{f(t)}\, dt$$

for all $x > 0$. Find the function $f(x)$ explicitly and verify the function you found satisfies the condition.

Solution: Clearly, $f(0) = 1$ and by differentiating with respect to x, $f'(x) = \sqrt{f(x)}$. This is a separable differential equation: $\frac{df(x)}{f^{1/2}(x)} = dx$ whose solution is $2f^{1/2}(x) = x + C$. $f(0) = 1$ implies $C = 2$ and $f(x) = \left(1 + \frac{x}{2}\right)^2$.

Verification is straightforward:

$$1 + \int_0^x 1 + \frac{t}{2}\, dt = 1 + x + \frac{x^2}{4} = (1 + \frac{x}{2})^2.$$

Alternatively, differentiating both sides of the equation gives $f'(x) = \sqrt{f(x)}$. Since f is differentiable, and $f(x) > 0$ for all x, we conclude that f' is differentiable everywhere, and differentiating again gives $f''(x) = \frac{1}{2}f'(x)/\sqrt{f(x)} = 1/2$ so $f''(x) = \frac{1}{4}x^2 + cx + d$ for some constants c and d. It follows that $f'(0) = c$ and $f(0) = d$. But from the original equation we see that $f(0) = 1$ and since $f'(0) = \sqrt{f(0)}$ we obtain $c = 1$. Thus $f(x) = \frac{1}{4}x^2 + x + 1$.

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6. Prove: If $a_1, a_2, a_3, \ldots$ is a positive sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent, then the series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is also convergent. Also, show that the converse is true under the additional assumption that the sequence (a_n) is monotone.

Solution: By the Cauchy-Schwarz inequality $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} \leq \left(\sum_{n=1}^{\infty} a_n\right)^{1/2} \left(\sum_{n=1}^{\infty} a_{n+1}\right)^{1/2}$ and each series on the right is convergent.

Now suppose $a_n \geq a_{n+1} > 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ is convergent. For any $N \geq 1$,

$$\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}} \quad \geq \quad \sum_{n=1}^{N} \sqrt{a_n a_{n+1}} \geq \sum_{n=1}^{N} \sqrt{a_{n+1} a_{n+1}} = \sum_{n=1}^{N} a_{n+1} = \sum_{n=2}^{N} a_n.$$

Therefore, $\sum_{n=1}^{N} a_n$ is a bounded monotone sequence of partial sums which implies $\sum_{n=1}^{\infty} a_n$ is convergent.

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7. Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with real coefficients, and let M denote its companion matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ denote the roots of p counted with multiplicity, and define the Vandermonde matrix to be

$$V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix}.$$

Show that $MV = VD$ where D is the diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$.

Solution: Let

$$v^{(i)} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix}.$$

denote the i-th column of V. Direct calculation yields

$$Mv^{(i)} = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ -c_0 - c_1\lambda_i - c_2\lambda_i^2 - \cdots - c_{n-1}\lambda_i^{n-1} \end{bmatrix} . = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ \lambda_i^n - p(\lambda_i) \end{bmatrix} . = \begin{bmatrix} \lambda_i \\ \lambda_i^2 \\ \lambda_i^3 \\ \vdots \\ \lambda_i^{n-1} \\ \lambda_i^n \end{bmatrix} = \lambda_i v^{(i)}$$

and the result follows easily.

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8. Let X_1 and X_2 be independent $Uniform(0,1)$ random variables. Let $Y = \max(X_1, X_2)$ and $Z = \min(X_1, X_2)$. Find the conditional density of Y given $Z = z$.

Solution: The joint distribution of Z and Y is given by

$$\begin{aligned} F(z,y) &= 2P[X_1 < X_2, X_1 < z, X_2 < y] \\ &= 2\int_0^y \int_0^{\min(x_2,z)} dx_1 dx_2 \\ &= 2\int_0^y \min(x_2, z) dx_2 \end{aligned}$$

for $0 \leq z, y \leq 1$, and the joint density of Z and Y is given by

$$f(z,y) = 2I\{0 < z \leq y < 1\}.$$

The marginal density of Z is given by

$$f_Z(z) = \int_z^1 2dy = 2(1-z)$$

for $z \in (0,1)$ and 0 otherwise. Thus the conditional density of Y given $Z = z$ is uniform on the interval $(z, 1)$.

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9. Let X_0 denote the high temperature in Baltimore on New Year's Day next year. For $n > 0$, let X_n denote the high temperature in Baltimore on New Year's Day n years later. Suppose the $\{X_n\}_{n=0}^{\infty}$ are independent and identically distributed random variables with a continuous distribution function. Let N denote the smallest number of years that elapse before there is a higher temperature than X_0 on New Year's Day in Baltimore.

[a] Find $P[N > n]$ for each $n = 1, 2, 3, \ldots$

[b] Find $E[N]$.

Solution: [a] N is the minimum n for which $X_n > X_0$. By symmetry and the fact that there is probability zero of any ties among the $\{X_i\}$s,

$$P[N > n] = P[X_n \leq X_0, X_{n-1} \leq X_0, X_{n-2} \leq X_0, \ldots, X_1 > \leq X_0] = \frac{1}{n+1}$$

for $n \geq 0$.

[b] Using the tail probability sum formula for the expectation of a non-negative integer-valued random variable,

$$E[N] = \sum_{n=0}^{\infty} P[N > n] = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty.$$

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10. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}$$

on the real line for $n = 1, 2, 3, \ldots$. Show that $\{f_n\}$ converges uniformly to a differentiable function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Solution: Since $2\sqrt{n}|x| \leq 1 + nx^2$,

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}}$$

and thus f_n converges uniformly to $f(x) \equiv 0$ as $n \to \infty$. A simple calculation yields

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} = \frac{1}{n} \frac{\frac{1}{n} - x^2}{(\frac{1}{n} + x^2)^2}.$$

Since $\lim_{n\to\infty} \frac{\frac{1}{n} - x^2}{(\frac{1}{n} + x^2)^2} = -\frac{1}{x^2}$ for $x \neq 0$, it follows that

$$\lim_{n\to\infty} f_n'(x) = 0 = f'(x)$$

if $x \neq 0$. However,

$$f_n'(0) = \frac{1 - n \cdot 0}{(1 + n \cdot 0)^2} = 1$$

for all n and thus

$$\lim_{n\to\infty} f_n'(0) = 1 \neq 0 = f'(0).$$

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11. Let $f : [-1, 1] \to R$ be three times differentiable and let $f(-1) = f(0) = 0$, $f(1) = 1$ and $f'(0) = 0$. Show that there exists $c \in (-1, 1)$ such that $f'''(c) \geq 3$.

Solution: It follows from Taylor's formula that there exist $x_1 \in (-1, 0)$ and $x_2 \in (0, 1)$ such that

$$0 = f(-1) = \frac{1}{2} f''(0) - \frac{1}{3!} f'''(x_1)$$
$$1 = f(1) = \frac{1}{2} f''(0) + \frac{1}{3!} f'''(x_2).$$

Thus

$$f'''(x_1) + f'''(x_2) = 6.$$

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12. Consider the following dice game: for each roll, you are paid the face value (the number that shows up). If a roll gives an odd number, then the game stops. If the roll gives an even number, you get paid the face value and then you flip a fair coin. If the coin shows up head (with probability 1/2), then you roll the die again and continue the game; otherwise the game stops. What is the expected payoff of this game.

Solution: Let X be the random payoff of the game, and Y be the outcome of your first throw. Conditioned on that Y is odd, the expected payoff is 3. If Y turns out to be even, then you get an expected face value of 4 plus the 50% chance of an extra throw. Consequently, we have $E[X|Y \text{ is even}] = 4 + \frac{1}{2}E[X]$. By the repeated expectation, the expected payoff satisfies the equation

$$E[X] = E[E[X|Y]] = \frac{1}{2}3 + \frac{1}{2}(4 + \frac{1}{2}E[X]),$$

which yields $E[X] = \frac{14}{3}$. (It's a 30% relative increase compared to the expected face value of a single roll).

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13. Let u and v be nonzero vectors in $\mathbb{R}^n$ whose inner product is nonzero, and let $\gamma \in \mathbb{R}$ be a nonzero scalar. Compute the two *distinct* eigenvalues of the $n \times n$ matrix A defined by

$$A = I + \gamma u v^T,$$

where I is the $n \times n$ identify matrix.

Solution: Let W be the $(n-1)$ dimensional space that is orthogonal to $V = Span(v)$. Let $\{w_1, w_2, \ldots, w_{n-1}\}$ be a basis for W so that $v^T w_j = 0$ for all $1 \leq j \leq n-1$. It follows that

$$Aw_j = (I + \gamma u v^T)w_j = w_j + \gamma(v^T w_j)u = w_j \quad \text{for all } 1 \leq j \leq n-1.$$

Therefore, A has $(n-1)$ eigenvalues equal to 1.

We now search for the remaining distinct eigenvalue. Since the eigenspace associated with the eigenvalue 1 contains at least the space W, we search for an eigen-pair (x, λ) such that $x \notin W$ and thus $v^T x \neq 0$. Since x is an eigenvector, it must satisfy

$$Ax = x + \gamma(v^T x)u = \lambda x$$

which implies that

$$(1-\lambda)x = -\gamma(v^T x)u. \tag{3}$$

The right-hand side of (3) is not zero since $u \neq 0$ by assumption and $v^T x \neq 0$ from above, so may may conclude that $\lambda \neq 1$. Combining this with the fact that eigen-vectors are nonzero by definition, we may further deduce from (3) that

$$x = \alpha u \quad \text{for some } \alpha \neq 0.$$

If we then plug this into (3), we have

$$\alpha(1-\lambda)u = -\alpha\gamma(v^T u)u \quad \Longrightarrow \quad (1-\lambda)u = -\gamma(v^T u)u$$

and since $u \neq 0$ by assumption, that

$$(1-\lambda) = -\gamma(v^T u) \quad \Longrightarrow \quad \lambda = 1 + \gamma(v^T u).$$

Thus, the second distinct eigenvalue of A is $\lambda = 1 + \gamma(v^T u)$.

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14. Let $n \geq 2$ be an integer and let S be the set of pairs (l, m), where $l, m = 1, \ldots, n$, with $l < m$. Consider the $\binom{n}{2} \times n$ matrix A defined by

$$A_{(l,m),j} = \begin{cases} 1 & \text{if } j = l \text{ or } j = m \\ 0 & \text{otherwise} \end{cases}, \quad (l,m) \in S, \quad j = 1, \ldots, n.$$

Compute the rank of A, for $n \geq 2$.

Solution: When $n = 2$, the matrix $A = [1, 1]$, hence the rank is 1.

Let us now assume $n \geq 3$. Then $\frac{n(n-1)}{2} > n$, hence $\operatorname{rank} A \leq n$.

It is easy to see that $A_{(1,2:n),2:n} = I_{n-1}$, the $(n-1) \times (n-1)$ identity matrix. Hence the 2nd through the last column vectors of A are linearly independent, which implies $\operatorname{rank} A \geq n - 1$.

The first column vector of A is $v_1 := [1, \ldots, 1, 0, \ldots, 0]^T$ where the first $n - 1$ entries are all one, and the rest are all zero. We show that the vector v_1 cannot be spanned by the rest of column vectors $v_2, \ldots, v_n$, and conclude that $\operatorname{rank} A = n$.

Suppose that there exists $c_2, \ldots, c_n$ such that $v_1 = c_2 v_2 + \cdots + c_n v_n$. Then using the observation $A_{(1,2:n),2:n} = I_{n-1}$ again, we conclude that $c_2 = \cdots = c_n = 1$. But then, the nth through the last entries of $c_2 v_2 + \cdots + c_n v_n$ are all two from the definition of A, and these do not match with the nth through the last entries of v_1, which are all zero. Therefore, v_1 cannot be spanned by the vectors $v_2, \ldots, v_n$.

15. Suppose that n points are independently chosen uniformly at random along a circle. Find the probability that they all lie in some semicircle, that is, that there is a line passing through the center of the disk enclosed by the circle such that all the points are on one side of the line.

Solution: This is derived from a Problem in Chapter 6 of Ross. Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and, for $i = 1, \ldots, n$, let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for 180°. Then A is the disjoint union of the events A_i, and clearly $P(A_i) = (1/2)^{n-1}$. Thus

$$P(A) = n(1/2)^{n-1}.$$