

Department of Applied Mathematics and Statistics  
The Johns Hopkins University

INTRODUCTORY EXAMINATION—SUMMER SESSION

Wednesday, August 28, 2013

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e.,  $2/3$  of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let  $x, y, z$  be elements of  $\mathbb{R}^n$ . Prove that the Euclidean norm  $\|\cdot\|$  satisfies

$$\left| \|x - z\| - \|y - z\| \right| \leq \|x - y\|.$$

2. Assume that  $A$  and  $B$  respectively are  $m$  by  $n$  and  $n$  by  $m$  real matrices. Prove that the non-vanishing eigenvalues of  $AB$  coincide with the non-vanishing eigenvalues of  $BA$  with equal multiplicity.

3. Let  $X$  and  $Y$  be independent random variables, each following a Laplace distribution with probability density function

$$f(t) = \frac{1}{2}e^{-|t|}.$$

Prove that, if  $aX + bY$  and  $cX + dY$  have the same distribution (with  $a, b, c, d \in \mathbb{R}$ ), then  $(a^2, b^2) = (c^2, d^2)$  or  $(a^2, b^2) = (d^2, c^2)$ .

4. Let  $A = [a_{ij}]$  be an  $n \times n$  matrix whose diagonal entries are all nonzero, and suppose that  $a_{ij} = 0$  if  $j \notin \{i, i + 1\}$ . Find and prove an explicit formula for the entries  $b_{ij}$  of the inverse matrix  $B = [b_{ij}]$ .

5. Suppose  $f : [0, \infty) \rightarrow \mathbb{R}$  is a positive and continuously differentiable function that satisfies the condition:

$$f(x) = 1 + \int_0^x \sqrt{f(t)} dt$$

for all  $x > 0$ . Find the function  $f(x)$  explicitly and verify the function you found satisfies the condition.

6. Prove: If  $a_1, a_2, a_3, \dots$  is a positive sequence of real numbers such that the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then the series  $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$  is also convergent. Also, show that the converse is true under the additional assumption that the sequence  $(a_n)$  is monotone.

7. Let  $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$  be a polynomial with real coefficients, and let  $M$  denote its companion matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-2} & -c_{n-1} \end{bmatrix}.$$

Let  $\lambda_1, \dots, \lambda_n$  denote the roots of  $p$  counted with multiplicity, and define the Vandermonde matrix to be

$$V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix}.$$

Show that  $MV = VD$  where  $D$  is the diagonal matrix whose diagonal entries are  $\lambda_1, \dots, \lambda_n$ .

8. Let  $X_1$  and  $X_2$  be independent  $Uniform(0, 1)$  random variables. Let  $Y = \max(X_1, X_2)$  and  $Z = \min(X_1, X_2)$ . Find the conditional density of  $Y$  given  $Z = z$ .
9. Let  $X_0$  denote the high temperature in Baltimore on New Year's Day next year. For  $n > 0$ , let  $X_n$  denote the high temperature in Baltimore on New Year's Day  $n$  years later. Suppose the  $\{X_n\}_{n=0}^\infty$  are independent and identically distributed random variables with a continuous distribution function. Let  $N$  denote the smallest number of years that elapse before there is a higher temperature than  $X_0$  on New Year's Day in Baltimore.

[a] Find  $P[N > n]$  for each  $n = 1, 2, 3, \dots$

[b] Find  $E[N]$ .

10. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}$$

on the real line for  $n = 1, 2, 3, \dots$ . Show that  $\{f_n\}$  converges uniformly to a differentiable function  $f$ , and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if  $x \neq 0$ , but false if  $x = 0$ .

11. Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be three times differentiable and let  $f(-1) = f(0) = 0$ ,  $f(1) = 1$  and  $f'(0) = 0$ . Show that there exists  $c \in (-1, 1)$  such that  $f'''(c) \geq 3$ .

12. Consider the following dice game: for each roll, you are paid the face value (the number that shows up). If a roll gives an odd number, then the game stops. If the roll gives an even number, you get paid the face value and then you flip a fair coin. If the coin shows up head (with probability  $1/2$ ), then you roll the die again and continue the game; otherwise the game stops. What is the expected payoff of this game.

13. Let  $u$  and  $v$  be nonzero vectors in  $\mathbb{R}^n$  whose inner product is nonzero, and let  $\gamma \in \mathbb{R}$  be a nonzero scalar. Compute the two *distinct* eigenvalues of the  $n \times n$  matrix  $A$  defined by

$$A = I + \gamma uv^T,$$

where  $I$  is the  $n \times n$  identity matrix.

14. Let  $n \geq 2$  be an integer and let  $S$  be the set of pairs  $(l, m)$ , where  $l, m = 1, \dots, n$ , with  $l < m$ . Consider the  $\binom{n}{2} \times n$  matrix  $A$  defined by

$$A_{(l,m),j} = \begin{cases} 1 & \text{if } j = l \text{ or } j = m \\ 0 & \text{otherwise} \end{cases}, \quad (l, m) \in S, \quad j = 1, \dots, n.$$

Compute the rank of  $A$ , for  $n \geq 2$ .

15. Suppose that  $n$  points are independently chosen uniformly at random along a circle. Find the probability that they all lie in some semicircle, that is, that there is a line passing through the center of the disk enclosed by the circle such that all the points are on one side of the line.