Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Tuesday, August 24, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Compute the following derivative and simplify as much as possible (restrict your attention to $x > 0$):

$$\frac{d}{dx}\int_{-x}^{x}\frac{1-e^{-xy}}{y}\,dy.$$

Solution: The singularity at $y = 0$ is removable. Apply the Liebniz rule:

$$\begin{aligned}\frac{d}{dx}\int_{-x}^{x}\frac{1-e^{-xy}}{y}\,dy &= \frac{1-e^{-x^2}}{x} - \frac{1-e^{x^2}}{-x}\cdot(-1) + \int_{-x}^{x}\frac{\partial}{\partial x}\frac{1-e^{-xy}}{y}\,dy \\ &= \frac{e^{x^2}-e^{-x^2}}{x} + \int_{-x}^{x} e^{-xy}\,dy \\ &= \frac{e^{x^2}-e^{-x^2}}{x} + \frac{e^{-xy}}{-x}\Big|_{y=-x}^{y=x} = 2\Big(\frac{e^{x^2}-e^{-x^2}}{x}\Big) \quad \text{or} \quad \frac{4\sinh(x^2)}{x}.\end{aligned}$$

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2. Consider the sequence (x_n) defined by $x_{n+1} = \sqrt{4x_n - 3}$, for $n \geq 1$ and $x_1 = 7$. Show that x_n is monotone and find its limit.

Solution: From the definition, $x_1 = 7 \geq 3$. Moreover, if, for some $k \geq 1$, $x_k \geq 3$, it follows that $x_{k+1} = \sqrt{4x_k - 3} \geq \sqrt{4(3) - 3} = 3$. Therefore, inductively, we see $x_n \geq 3$ for all $n \geq 1$. Now, $x_{n+1}^2 - x_n^2 = 4x_n - 3 - x_n^2 = -(x_n - 1)(x_n - 3) \leq 0$ since $x_n \geq 3$ for all n. Consequently, $x_{n+1}^2 \leq x_n^2$ for all n which in turn implies $x_{n+1} \leq x_n$, i.e., (x_n) is monotone decreasing. Since the sequence is also bounded below (indeed, by 3) the sequence converges. Let $x = \lim_{n\to\infty} x_n$. We know $x \geq 3$ already. After taking limits in the recursive expression, we find $x^2 - 4x + 3 = 0$ and $x = 1$ or $x = 3$. The limit is, therefore, 3.

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3. A subset S of $\mathbb{R}$ is said to be *disconnected* if there exist disjoint open sets U and V in $\mathbb{R}$ such that $S \subseteq U \cup V$, $S \cap U \neq \varnothing$, and $S \cap V \neq \varnothing$. If S is not disconnected, then it is said to be *connected.* Suppose S is connected and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove: $f(S)$ is connected.

Solution: Suppose S is connected and $f : \mathbb{R} \to \mathbb{R}$ is continuous, and, for the sake of contradiction, $f(S)$ is disconnected. Then there are disjoint open sets U and V of $\mathbb{R}$ such that $f(S) \subseteq U \cup V$, $f(S) \cap U \neq \varnothing$, and $f(S) \cap V \neq \varnothing$. Therefore, $S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Since f is continuous, the disjoint sets $f^{-1}(U)$ and $f^{-1}(V)$ are open; further, $S \cap f^{-1}(U) \neq \varnothing$ and $S \cap f^{-1}(V) \neq \varnothing$. Thus S is disconnected, which is a contradiction.

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4. Show that the sequence $f_n(x) = \frac{x}{1+nx^2}$, $n = 1, 2, 3, \ldots$ converges uniformly to some function $f(x)$ on $\mathbb{R}$.

Solution: For each fixed $x \in \mathbb{R}$ it is clear the sequence $(f_n(x))$ converges to zero. Let $\varepsilon > 0$ be given.

For each n note that $f_n(x)$ is continuously differentiable and $\lim_{|x|\to\infty} |f_n(x)| = \lim_{|x|\to\infty} \frac{|x|}{1+nx^2} = 0$. Moreover, $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = 0$ exactly when $x = \pm\frac{1}{\sqrt{n}}$. Therefore, $\max_{x\in\mathbb{R}} |f_n(x)| = |f_n(\frac{1}{\sqrt{n}})| = \frac{1}{2\sqrt{n}}$. Now, for any $x \in \mathbb{R}$, $|f_n(x)| \leq \max_y |f_n(y)| = \frac{1}{2\sqrt{n}} < \varepsilon$ when $n > \frac{1}{4\varepsilon^2}$. Since this choice of n does not depend on x the convergence to 0 is uniform.

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5. Prove or disprove: For all $x, y \leq 0$ we have $|e^x - e^y| \leq |x - y|$.

Solution: The claim is true. Fix $y \leq 0$ and consider the function $g(x) = e^x$ for $x \leq 0$. Apply the mean value theorem to the function g:

$$e^x - e^y = e^\xi (x - y),$$

where ξ is between x and y. Now, since $x \leq 0$ and $y \leq 0$, we have $e^\xi \leq 1$; consequently,

$$|e^x - e^y| = |e^\xi (x - y)| \leq |x - y|$$

and the result follows.

Alternate solution: $|e^y - e^x| = \left|\int_x^y e^u \, du\right| \leq \left|\int_x^y e^0 \, du\right| = |x - y|$ when $x, y \leq 0$.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
AFTERNOON EXAM–PROBABILITY

Tuesday, August 24, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A spinner has 4 equally likely regions, and a person spins this spinner repeatedly. Let N be the number of spins required for every region to be seen at least once. Compute $E(N)$.

Solution: For $i = 1, 2, 3, 4$, let X_i be the number of additional spins necessary to land in a region other than the previous $i - 1$ distinct regions. Clearly, $X_1 \equiv 1$ since the very first spin will land in one of the four regions. Since trials are independent, the number X_2 of additional spins necessary to land in a region different from this first region is a geometric rv with success probability $3/4$. Likewise, the number X_3 of additional spins necessary to land in a region different from the first two is geometric with parameter $2/4$, and the number X_4 of additional spins necessary to land in the last possible region is geometric with parameter $1/4$. Now, $N = X_1 + X_2 + X_3 + X_4$, from which we have $E(N) = E(X_1) + E(X_2) + E(X_3) + E(X_4) = 1 + \frac{4}{3} + \frac{4}{2} + \frac{4}{1} = \frac{25}{3}$.

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2. 106 students in a probability class were assigned to each toss a fair coin 100 times. Compute the probability that no student tosses exactly 50 heads. Do not simplify.

Solution: The probability a student does not toss exactly 50 heads is $1 - \binom{100}{50}\left(\frac{1}{2}\right)^{100}$. Therefore, the probability that all students do not toss exactly 50 heads is

$$\left[1 - \binom{100}{50}\left(\frac{1}{2}\right)^{100}\right]^{106}.$$

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3. Consider two sequences of random variables $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$. Suppose X_n converges to a with probability 1 and Y_n converges to b with probability 1. Prove $X_n + Y_n$ converges to $a + b$ with probability 1.

Solution: Let $A = \{\omega \in \Omega : X_n(\omega) \text{ does not converge to } a\}$ and similarly $B = \{\omega \in \Omega : Y_n(\omega) \text{ does not converge to } b\}$. By supposition, $P(A) = P(B) = 0$. Define $C = \{\omega \in \Omega : X_n(\omega) + Y_n(\omega) \text{ does not converge to } a+b\}$. Note $C \subseteq A \cup B$. Therefore, $P(C) \leq P(A \cup B) \leq P(A) + P(B) = 0$ implies $P(C) = 0$. Consequently, $X_n + Y_n$ converges to $a + b$ with probability 1.

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4. Suppose $X_1, X_2, X_3, \dots$ is an i.i.d. sequence such that X_1 has the moment generating function $M(\theta) = \frac{1}{1-\theta^2}$ for $|\theta| < 1$. For each integer $n \geq 1$, let

$$S_n = \sum_{i=1}^{n} X_i^2.$$

If it is possible, find, with justification, sequences (a_n) and (b_n) such that $\frac{S_n - a_n}{b_n}$ converges in distribution to a standard normal. If it is not possible, explain why not.

Solution: If we set $Y_i = X_i^2$ then the sequence (Y_i) is i.i.d. and $E(Y_i) = M''(0)$ and $E(Y_i^2) = M^{(4)}(0)$. By writing $M(\theta) = \frac{1}{2}(1-\theta)^{-1} + \frac{1}{2}(1+\theta)^{-1}$ we can easily find the moments: $M''(0) = 2$ and $M^{(4)}(0) = 24$. Therefore, $E(Y_i) = 2$ and $\mathrm{Var}(Y_i) = 20$. It follows immediately from the Central Limit Theorem that by taking $a_n = 2n$ and $b_n = \sqrt{20n}$ we have the stated convergence.

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5. If X and Y are independent unit exponential random variables, i.e., each possess the pdf $f(x) = e^{-x}$ for $x > 0$, find the joint density of their polar coordinates (R, Θ) of the point (X, Y). Are R and Θ independent?

Solution: Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$ it follows that the Jacobian of the transformation taking (r, θ) to (x, y) is r and the joint pdf is

$$f_{R,\Theta}(r, \theta) = e^{-r\cos(\theta)} e^{-r\sin(\theta)} r = r e^{-r(\cos(\theta)+\sin(\theta))} \quad \text{for } 0 < r < \infty,\ 0 < \theta < \frac{\pi}{2}.$$

The random variables R and Θ are *not* independent as the joint pdf doesn't factor as a product of pdfs involving r and θ separately.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–LINEAR ALGEBRA

Wednesday, August 25, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 PM and end at 11:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let x and y be two vectors in $\mathbb{R}^n$. Find a value of $\alpha \in \mathbb{R}$ such that $\|x - \alpha y\|^2$ is a minimum and show that $\|x - \alpha y\|^2 + \|\alpha y\|^2 = \|x\|^2$.

Solution: We assume $y \neq \mathbf{0}$; otherwise, there is nothing to show.

Let $g(u) := \|x - uy\|^2 = \langle x - uy, x - uy\rangle = \|x\|^2 - 2u\langle x, y\rangle + u^2\|y\|^2$. Differentiating we have $g'(u) = -2\langle x, y\rangle + 2u\|y\|^2 = 0$ implies $\alpha = \langle x, y\rangle/\|y\|^2$ is the critical value, in fact, the minimizer. Moreover,

$$\begin{aligned}
\|x - \alpha y\|^2 + \|\alpha y\|^2 &= \|x\|^2 - 2\alpha\langle x, y\rangle + 2\alpha^2\|y\|^2 \\
&= \|x\|^2 - 2\frac{\langle x, y\rangle}{\|y\|^2}\langle x, y\rangle + 2\frac{\langle x, y\rangle^2}{\|y\|^4}\|y\|^2 \\
&= \|x\|^2,
\end{aligned}$$

which was to be shown.

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2. Let $A = [a_{ij}(t)]$ be a 2×2 matrix of real-valued differentiable functions. Show that

$$\frac{d}{dt}\det A = \det D_1 + \det D_2$$

where D_i is the matrix A with its ith column replaced by the column of the derivatives of the entries in the ith column of A.

Solution: By a direct calculation

$$\begin{aligned}
\frac{d}{dt}\det A &= \frac{d}{dt}\sum_\sigma \operatorname{sgn}(\sigma)a_{1\sigma_1}(t)a_{2\sigma_2}(t) \\
&= \frac{d}{dt}\Big(a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t)\Big) \\
&= a'_{11}(t)a_{22}(t) + a_{11}(t)a'_{22}(t) - \Big(a'_{12}(t)a_{21}(t) + a_{12}(t)a'_{21}(t)\Big) \\
&= \Big(a'_{11}(t)a_{22}(t) - a'_{12}(t)a_{21}(t)\Big) + \Big(a_{11}(t)a'_{22}(t) - a_{12}(t)a'_{21}(t)\Big) \\
&= \det D_1 + \det D_2.
\end{aligned}$$

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3. For any $\mathbf{x} \in \mathbb{R}^n$, show that the Householder matrix $M_{\mathbf{x}} = I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}$ is an orthogonal matrix and find its eigenvalues. Here, I is the $n \times n$ identity matrix, $\|x\|$ is the Euclidean (usual) norm, and T is transpose.

Solution: Clearly, $M_{\mathbf{x}}^{\mathrm{T}} = M_{\mathbf{x}}$. Furthermore,

$$\begin{aligned} M_{\mathbf{x}} M_{\mathbf{x}}^{\mathrm{T}} &= \left(I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}\right)\left(I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}\right)^{\mathrm{T}} \\ &= \left(I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}\right)\left(I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}\right) \\ &= I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}} - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}} + \frac{4}{\|\mathbf{x}\|^4}\mathbf{x}\left(\mathbf{x}^{\mathrm{T}}\mathbf{x}\right)\mathbf{x}^{\mathrm{T}} \\ &= I - \frac{4}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}} + \frac{4}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}} = I. \end{aligned}$$

Therefore, $M_{\mathbf{x}} M_{\mathbf{x}}^{\mathrm{T}} = M_{\mathbf{x}}^{\mathrm{T}} M_{\mathbf{x}} = M_{\mathbf{x}}^2 = I$ and $M_{\mathbf{x}}$ is an orthogonal matrix.

Moreover, since $M_{\mathbf{x}}$ is orthogonal, $M_{\mathbf{x}}\mathbf{v} = \lambda\mathbf{v}$ implies $M_{\mathbf{x}}^2\mathbf{v} = \lambda M_{\mathbf{x}}\mathbf{v} = \lambda^2\mathbf{v} = \mathbf{v}$.

Therefore, the eigenvalues are all ± 1.

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4. Let A and B be $n \times n$ matrices such that $AB = BA$. Prove: if all the eigenvalues of A are distinct, then each eigenvector of A is an eigenvector of B.

Solution: Since A has n distinct eigenvalues, the dimension of each eigenspace is 1.

Let λ be an eigenvalue of the matrix A and let x be the eigenvector corresponding to λ. Since the eigenspace E_λ for λ is one dimensional and $x \in E_\lambda$ is a nonzero vector in it, $\{x\}$ is a basis for E_λ. That is, we have $E_\lambda = \{cx | c \in \mathbb{C}\}$.

Now, $Ax = \lambda x$ implies $B(Ax) = \lambda(Bx)$. Since $AB = BA$ it follows that $A(Bx) = \lambda(Bx)$ and $Bx \in E_\lambda$. This implies $Bx = cx$ for some $c \in \mathbb{C}$. Therefore, the vector x is also an eigenvector corresponding to the eigenvalue c of the matrix B.

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5. Let A be a real $n \times n$ matrix and let $\rho(A)$ be its spectral radius.

   Prove or disprove: $\rho(A + B) \leq \rho(A) + \rho(B)$.

Solution: We disprove: Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\rho(A+B) = 1$ whereas $\rho(A) = \rho(B) = 0$.
