Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–REAL ANALYSIS

Tuesday, August 24, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Compute the following derivative and simplify as much as possible (restrict your attention to $x > 0$):

$$\frac{d}{dx} \int_{-x}^{x} \frac{1 - e^{-xy}}{y} \, dy.$$

2. Consider the sequence (x_n) defined by $x_{n+1} = \sqrt{4x_n - 3}$, for $n \geq 1$ and $x_1 = 7$.

   Show that x_n is monotone and find its limit.

3. A subset S of $\mathbb{R}$ is said to be *disconnected* if there exist disjoint open sets U and V in $\mathbb{R}$ such that $S \subseteq U \cup V$, $S \cap U \neq \varnothing$, and $S \cap V \neq \varnothing$. If S is not disconnected, then it is said to be *connected*. Suppose S is connected and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove: $f(S)$ is connected.

4. Show that the sequence $f_n(x) = \frac{x}{1+nx^2}$, $n = 1, 2, 3, \ldots$ converges uniformly to some function $f(x)$ on $\mathbb{R}$.

5. Prove or disprove: For all $x, y \leq 0$ we have $|e^x - e^y| \leq |x - y|$.

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Afternoon Exam–Probability

Tuesday, August 24, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A spinner has 4 equally likely regions, and a person spins this spinner repeatedly. Let N be the number of spins required for every region to be seen at least once. Compute $E(N)$.

2. 106 students in a probability class were assigned to each toss a fair coin 100 times. Compute the probability that no student tosses exactly 50 heads. Do not simplify.

3. Consider two sequences of random variables $X_1, X_2, X_3, \ldots$ and $Y_1, Y_2, Y_3, \ldots$. Suppose X_n converges to a with probability 1 and Y_n converges to b with probability 1. Prove $X_n + Y_n$ converges to $a + b$ with probability 1.

4. Suppose $X_1, X_2, X_3, \ldots$ is an i.i.d. sequence such that X_1 has the moment generating function $M(\theta) = \frac{1}{1-\theta^2}$ for $|\theta| < 1$. For each integer $n \geq 1$, let

$$S_n = \sum_{i=1}^{n} X_i^2.$$

If it is possible, find, with justification, sequences (a_n) and (b_n) such that $\frac{S_n - a_n}{b_n}$ converges in distribution to a standard normal. If it is not possible, explain why not.

5. If X and Y are independent unit exponential random variables, i.e., each possess the pdf $f(x) = e^{-x}$ for $x > 0$, find the joint density of their polar coordinates (R, Θ) of the point (X, Y). Are R and Θ independent?

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–LINEAR ALGEBRA

Wednesday, August 25, 2021

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 PM and end at 11:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let x and y be two vectors in $\mathbb{R}^n$. Find a value of $\alpha \in \mathbb{R}$ such that $\|x - \alpha y\|^2$ is a minimum and show that $\|x - \alpha y\|^2 + \|\alpha y\|^2 = \|x\|^2$.

2. Let $A = [a_{ij}(t)]$ be a 2×2 matrix of real-valued differentiable functions. Show that

$$\frac{d}{dt}\det A = \det D_1 + \det D_2$$

where D_i is the matrix A with its ith column replaced by the column of the derivatives of the entries in the ith column of A.

3. For any $\mathbf{x} \in \mathbb{R}^n$, show that the Householder matrix $M_{\mathbf{x}} = I - \frac{2}{\|\mathbf{x}\|^2}\mathbf{x}\mathbf{x}^{\mathrm{T}}$ is an orthogonal matrix and find its eigenvalues. Here, I is the $n \times n$ identity matrix, $\|x\|$ is the Euclidean (usual) norm, and T is transpose.

4. Let A and B be $n \times n$ matrices such that $AB = BA$. Prove: if all the eigenvalues of A are distinct, then each eigenvector of A is an eigenvector of B.

5. Let A be a real $n \times n$ matrix and let $\rho(A)$ be its spectral radius.

   Prove or disprove: $\rho(A + B) \leq \rho(A) + \rho(B)$.