Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Tuesday, August 25, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Evaluate the following limit (if it exists):
$$\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right).$$
Here x is tending to 0 through values in the interval $0 < x < \frac{\pi}{2}$.

Solution: Since we have an indeterminant form, applications of L'Hôpital's rule gives
$$\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) = \lim_{x\to 0+}\frac{\sin(x)-x}{x\sin(x)} = \lim_{x\to 0+}\frac{\cos(x)-1}{x\cos(x)+\sin(x)}$$
$$= \lim_{x\to 0+}\frac{-\sin(x)}{2\cos(x)-x\sin(x)} = \frac{0}{2} = 0.$$
Alternative solution. Since $x-\frac{x^3}{3} < \sin(x) < x$ for $0 < x < \frac{\pi}{2}$, it follows that
$$0 < \frac{1}{x}-\frac{1}{\sin(x)} < \frac{1}{x}-\frac{1}{x-\frac{x^3}{3}} = \frac{-x}{3(1-\frac{x^2}{3})}.$$
Consequently, $\frac{1}{x}-\frac{1}{\sin(x)}$ tends to 0 as x tends to 0.

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2. Let $A \subseteq \mathbb{R}^2$ be open. Prove: for each $x \in \mathbb{R}$, the set $A_x = \{y : (x, y) \in A\}$ is an open subset of $\mathbb{R}$.

Solution: Let A be an open subset of $\mathbb{R}^2$. Fix an $x \in \mathbb{R}$ for which A_x is nonempty (otherwise there is nothing to show); and let $y \in A_x$ be arbitrarily chosen. Since (x, y) belongs to A, there exists an open rectangle $(a_1, a_2) \times (b_1, b_2)$ entirely contained in A centered at (x, y). In particular, we found an open interval (b_1, b_2) containing y, and A_x is an open subset of $\mathbb{R}$.

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3. For each integer $n \geq 1$ define $f_n : [0, 2] \to \mathbb{R}$ by $f_n(x) = \frac{x^n}{1+x^n}$. Prove or disprove that these functions converge uniformly on $[0, 2]$.

Solution: These functions do *not* converge uniformly on $[0, 2]$, because, if they did converge uniformly on the compact interval $[0, 2]$ the limit function would necessarily be continuous. However, for $0 \leq x < 1$, $\lim_{n\to\infty} f_n(x) = 0$, whereas $f_n(1) = \frac{1}{2}$ for all n. This shows our limit function is *discontinuous* (at $x = 1$).

4. Give an example of a uniformly continuous function on $[0,1]$ that is differentiable on $(0,1)$ but whose derivative is not bounded on $(0,1)$. Be sure to justify your claim.

Solution: Consider the function $f(x) = x\sin(\frac{1}{x})$ for $0 < x \leq 1$, and $f(0) = 0$. On the open interval $(0,1)$, both $\sin(\frac{1}{x})$ and x are continuously differentiable functions, and, therefore, their product is differentiable. Moreover, since $|\sin(\frac{1}{x})| \leq 1$ for all $0 < x \leq 1$, we have $\lim_{x\to 0+} x\sin(\frac{1}{x}) = 0 = f(0)$ and our $f(x)$ is continuous on $[0,1]$ and consequently uniformly continuous. However, for $x > 0$, $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$ is unbounded as x approaches 0.

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5. Let $F(x) = (f(x) - f(a))(g(b) - g(x))$, where f and g are continuous on $[a,b]$ and differentiable on (a,b). Suppose further that $g'(x)$ is never zero. Show that there must exist ξ between a and b such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(\xi) - f(a)}{g(b) - g(\xi)}.$$

Solution: $F(x)$ is also differentiable when $a < x < b$ and

$$F'(x) = f'(x)(g(b) - g(x)) - g'(x)(f(x) - f(a)).$$

Now, $F(a) = F(b) = 0$ and it follows by the Mean–Value Theorem that there exists a ξ between a and b such that $F'(\xi)(b-a) = F(b) - F(a) = 0$ implying $F'(\xi) = 0$. That is,

$$f'(\xi)(g(b) - g(\xi)) - g'(\xi)(f(\xi) - f(a)) = 0$$

or, after rearranging terms,

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(\xi) - f(a)}{g(b) - g(\xi)}.$$

The division by $g'(\xi)$ is justified since it is never zero; moreover, $g(b) - g(\xi) \neq 0$ because, if it did, the Mean–Value Theorem would imply $g'(c) = 0$ for some $c \in (\xi, b)$.

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Afternoon Exam–Probability

Tuesday, August 25, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let Z be a standard normal random variable. Find the pdf of $Y = |Z|$ and the mean and variance of Y.

Solution: If $y < 0$, the cdf of Y is $F_Y(y) = 0$. If $y \geq 0$, then

$$F_Y(y) = P(Y \leq y) = P(-y \leq Z \leq y) = \Phi(y) - \Phi(-y) = 2\Phi(y) - 1,$$

where Φ is the cdf of Z. Therefore, the pdf of Y is $f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-y^2/2} & y \geq 0 \\ 0 & y < 0 \end{cases}$.

$E(Y) = \int_0^\infty y \cdot \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\, dy = \sqrt{\frac{2}{\pi}} \int_0^\infty y e^{-y^2/2}\, dy = \sqrt{\frac{2}{\pi}}$
$E(Y^2) = E(Z^2) = \mathrm{Var}(Z) + \{E(Z)\}^2 = 1.$
$\mathrm{Var}(Y) = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}.$

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2. A fair coin is tossed repeatedly. Let X represent the trial on which the first head occurs. Compute the probability X is divisible by 2 or 3.

Solution:
$P(X \text{ is divisible by } 2) = \sum_{j=1}^\infty P(X = 2j) = \sum_{j=1}^\infty (\frac{1}{2})^{2j} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$
$P(X \text{ is divisible by } 3) = \sum_{j=1}^\infty P(X = 3j) = \sum_{j=1}^\infty (\frac{1}{2})^{3j} = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$
$P(X \text{ is divisible by } 6) = \sum_{j=1}^\infty P(X = 6j) = \sum_{j=1}^\infty (\frac{1}{2})^{6j} = \frac{\frac{1}{64}}{1 - \frac{1}{64}} = \frac{1}{63}.$
$P(X \text{ is divisible by 2 or 3}) = \frac{1}{3} + \frac{1}{7} - \frac{1}{63} = \frac{29}{63}.$

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3. For a positive integer $n \geq 1$ let $N_n = \{1, 2, \ldots, n\}$ and consider the power set 2^{N_n} of N_n, i.e., the set of all subsets of N_n. An experiment has us select $A, B \in 2^{N_n}$ uniformly at random *with* replacement (so $A = B$ is possible). Compute the probability that one is a subset of the other.

Solution: First, we compute the probability A is a subset of B. All ordered pairs (A, B) have probability $1/2^{2n}$. Moreover, for each integer $j = 0, 1, \ldots, n$, if $|A| = j$, then there are 2^{n-j} subsets B of 2^{N_n} that contain A. Moreover, there are $\binom{n}{j}$ such subsets A having cardinality j. Therefore, the probability A is a subset of B is $\frac{\sum_{j=0}^{n}\binom{n}{j}2^{n-j}}{2^{2n}} = \frac{3^n}{2^{2n}}$.

By a symmetric argument, the probability B is a subset of A also equals $\frac{3^n}{2^{2n}}$.

The probability $A = B$ is $\frac{2^n}{2^{2n}}$.

Finally, the probability one is a subset of the other is $2 \cdot \frac{3^n}{2^{2n}} - \frac{2^n}{2^{2n}} = \frac{3^n - 2^{n-1}}{2^{2n-1}}$.

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4. Suppose $X_1, X_2, X_3, \dots$ is a sequence of pairwise *uncorrelated* random variables having finite mean μ and finite variance σ^2. For each integer $n \geq 1$, let $S_n = \sum_{j=1}^{n} X_j$. Prove that $\frac{S_n}{n}$ converges to μ in probability.

Solution: Since $E(\frac{S_n}{n}) = \mu$ we have (using bilinearity of covariance)

$$\begin{aligned}
\text{Var}(\frac{S_n}{n}) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j) \\
&= \frac{1}{n^2}\sum_{\substack{i=1 \\ i=j}}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j) + \frac{1}{n^2}\sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j) \\
&= \frac{1}{n^2}\sum_{\substack{i=1 \\ i=j}}^{n}\sum_{j=1}^{n}\sigma^2 + \frac{1}{n^2}\sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n}0 \\
&= \frac{\sigma^2}{n}.
\end{aligned}$$

The Chebyshev inequality now implies that, for any $\varepsilon > 0$,

$$P(\left|\frac{S_n}{n} - \mu\right| > \varepsilon) \leq \frac{\text{Var}(\frac{S_n}{n})}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0$$

as $n \to \infty$.

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5. There are ten (10) boys standing in a circle which includes Fred. Suppose six (6) girls enter the circle to form a larger circle with each girl between two boys. If all such

ways of forming a larger circle are equally likely, what's the probability that Fred remains between two boys?

Solution: The ten boys make ten adjacencies for the girls. There are $\binom{10}{6}$ equally likely ways the girls can create a larger circle with each girl between two boys. If Fred is to remain between two boys we lose two adjacencies for the girls to enter, and, therefore, there are $\binom{8}{6}$ ways the girls can create a larger circle and keep Fred between two boys. The desired probability is

$$\frac{\binom{8}{6}}{\binom{10}{6}} = \frac{2}{15}.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Evening Exam–Linear Algebra

Wednesday, August 26, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

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4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 PM and end at 11:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Find the value(s) of λ for which the nonhomogeneous linear system

$$\begin{aligned} 5x_1 + 2x_2 - \lambda x_1 &= 4 \\ 2x_1 + 2x_2 - \lambda x_2 &= 7 \end{aligned}$$

has a solution and write this solution as a function of λ.

Solution: By setting $A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ and $c = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ we can write the system as $Ax - \lambda x = c$. The matrix A has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$ with respective eigenvectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. There is no solution to the nonhomogeneous equation if $\lambda = 6$ or 1. When $\lambda \neq 6, 1$, the solution can be written as a linear combination of the eigenvectors: $x = \alpha_1 v_1 + \alpha_2 v_2$.

Substituting this into our equation gives $A(\alpha_1 v_1 + \alpha_2 v_2) - \lambda(\alpha_1 v_1 + \alpha_2 v_2) = c$, or, equivalently,

$$\alpha_1(6 - \lambda)v_1 + \alpha_2(1 - \lambda)v_2 = c.$$

Dotting both sides separately by v_1 and then by v_2 gives the equations:

$$\alpha_1(6 - \lambda)5 = 15 \quad \text{and} \quad \alpha_2(1 - \lambda)5 = -10,$$

which gives $\alpha_1 = \frac{3}{6-\lambda}$ and $\alpha_2 = -\frac{2}{1-\lambda}$ and

$$x = \frac{3}{6 - \lambda}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{2}{1 - \lambda}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{(6 - \lambda)(1 - \lambda)}\begin{bmatrix} 6 - 4\lambda \\ 27 - 7\lambda \end{bmatrix}.$$

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2. Let A and B be complex $n \times n$ Hermitian matrices. Prove that AB is Hermitian if and only if A and B commute.

Solution: If $AB = BA$, then, since A and B are Hermitian, $(AB)^* = (BA)^* = A^*B^* = AB$ shows AB is also Hermitian.

Conversely, if $AB = (AB)^*$, then, since A and B are Hermitian, $AB = B^*A^* = BA$ shows A and B commute.

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3. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that there is a vector $v \in V$ such that the list

$$\{v, Tv, T^2v, \dots\}$$

spans V. Show that if $S : V \to V$ is a linear transformation that commutes with T, then there is a polynomial f such that $S = f(T)$.

Solution: We first show that if $n = \dim(V)$, then $\{v, Tv, T^2v, \dots, T^{n-1}v\}$ is a basis for V. To this end it suffices to show $\{v, Tv, T^2v, \dots, T^{n-1}v\}$ is linearly independent. For each $k \le n-1$, we show that $T^k v$ is not in the span of $\{v, Tv, \dots, T^{k-1}v\}$. To see why, note that if for some k, $T^k v$ was in the span of $\{v, Tv, \dots, T^{k-1}v\}$, then $T^k v = \sum_{j=1}^{k-1} c_j T^j v = \sum_{j=1}^{k-2} c_j T^j v + c_k T^{k-1} v$ and this would imply that $T^{k+1}v = T(T^k)v = \sum_{j=1}^{k-2} c_j T^{j+1} v + c_k T^k v$ would belong to the span of $\{v, Tv, \dots, T^{k-1}v\}$. Inductively, it would then follow that all $T^{k+i}v$ belongs to the span of $\{v, Tv, \dots, T^{k-1}v\}$ for all $i \ge 0$, which is a contradiction because then $\{v, Tv, \dots, T^{k-1}v\}$ spans V and the dimension of V is strictly less than n.

Now, since $\{v, Tv, \dots, T^{n-1}v\}$ forms a basis for V, there exist coefficients c_i such that $S(v) = \sum_{i=0}^{n-1} c_i T^i(v)$. Let $f(x) = \sum_{i=0}^{n-1} c_i x^i$. Then for each j, we have

$$\begin{aligned}
f(T)(T^j(v)) &= \Big(\sum_{i=0}^{n-1} c_i T^i\Big)(T^j(v)) = \sum_{i=0}^{n-1} c_i T^i \circ T^j(v) \\
&= \sum_{i=0}^{n-1} T^j(c_i T^i(v)) = T^j\Big(\sum_{i=0}^{n-1} c_i T^i(v)\Big) = T^j(S(v)) \\
&= S(T^j(v)).
\end{aligned}$$

This shows that $f(T)$ and S agree on the basis $\{v, Tv, \dots, T^{n-1}v\}$ of V, so $S = f(T)$.

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4. Consider the $n \times n$ matrix $A_n = [a_{ij}]$, where $a_{ii} = 1$ and, for all $i \neq j$ and some $-1 < \rho < 1$, $a_{ij} = \rho$. For example,

$$A_2 = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}.$$

Compute $\det(A_n)$ as a function of n.

Solution: Consider A_n. Adding the last $n-1$ rows of A_n to the first row of A_n gives the matrix:

$$\widetilde{A_n} = \begin{bmatrix} 1+(n-1)\rho & 1+(n-1)\rho & \cdots & 1+(n-1)\rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}.$$

Now, taking the first column of $\widetilde{A_n}$ and subtracting from each successive column gives the matrix

$$C = \begin{bmatrix} 1+(n-1)\rho & 0 & \cdots & 0 \\ \rho & 1-\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \rho & 0 & \cdots & 1-\rho \end{bmatrix}.$$

Moreover, $\det(A_n) = \det(C) = (1+(n-1)\rho)(1-\rho)^{n-1}$.

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5. Find a real matrix A such that $e^A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ or prove no such matrix can exist.

Solution: It can be shown that $e^{tJ} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Consequently, $e^{\pi J} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Also, it can be shown that $e^{\ln(2)I} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Now, since πJ and $\ln(2)I$ commute, it follows that

$$e^{\pi J+\ln(2)I} = e^{\pi J}e^{\ln(2)I} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Thus, the matrix

$$A = \pi J + \ln(2)I = \begin{bmatrix} \ln(2) & \pi \\ -\pi & \ln(2) \end{bmatrix}$$

is one such matrix with the desired property.

Note: If S is a 2×2 invertible matrix, then SAS^{-1} is another matrix with the desired property.
