Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–REAL ANALYSIS

Tuesday, August 25, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Evaluate the following limit (if it exists):

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right).$$

Here x is tending to 0 through values in the interval $0 < x < \frac{\pi}{2}$.

2. Let $A \subseteq \mathbb{R}^2$ be open. Prove: for each $x \in \mathbb{R}$, the set $A_x = \{y : (x, y) \in A\}$ is an open subset of $\mathbb{R}$.

3. For each integer $n \geq 1$ define $f_n : [0, 2] \to \mathbb{R}$ by $f_n(x) = \frac{x^n}{1+x^n}$. Prove or disprove that these functions converge uniformly on $[0, 2]$.

4. Give an example of a uniformly continuous function on $[0, 1]$ that is differentiable on $(0, 1)$ but whose derivative is not bounded on $(0, 1)$. Be sure to justify your claim.

5. Let $F(x) = (f(x) - f(a))(g(b) - g(x))$, where f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose further that $g'(x)$ is never zero. Show that there must exist ξ between a and b such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(\xi) - f(a)}{g(b) - g(\xi)}.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–PROBABILITY

Wednesday, August 26, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let Z be a standard normal random variable. Find the pdf of $Y = |Z|$ and the mean and variance of Y.

2. A fair coin is tossed repeatedly. Let X represent the trial on which the first head occurs. Compute the probability X is divisible by 2 or 3.

3. For a positive integer $n \geq 1$ let $N_n = \{1, 2, \ldots, n\}$ and consider the power set 2^{N_n} of N_n, i.e., the set of all subsets of N_n. An experiment has us select $A, B \in 2^{N_n}$ uniformly at random *with* replacement (so $A = B$ is possible). Compute the probability that one is a subset of the other.

4. Suppose $X_1, X_2, X_3, \ldots$ is a sequence of pairwise *uncorrelated* random variables having finite mean μ and finite variance σ^2. For each integer $n \geq 1$, let $S_n = \sum_{j=1}^{n} X_j$. Prove that $\frac{S_n}{n}$ converges to μ in probability.

5. There are ten (10) boys standing in a circle which includes Fred. Suppose six (6) girls enter the circle to form a larger circle with each girl between two boys. If all such ways of forming a larger circle are equally likely, what's the probability that Fred remains between two boys?

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Evening Exam–Linear Algebra

Wednesday, August 26, 2020

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 PM and end at 11:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Find the value(s) of λ for which the nonhomogeneous linear system

$$\begin{aligned} 5x_1 + 2x_2 - \lambda x_1 &= 4 \\ 2x_1 + 2x_2 - \lambda x_2 &= 7 \end{aligned}$$

has a solution and write this solution as a function of λ.

2. Let A and B be complex $n \times n$ Hermitian matrices. Prove that AB is Hermitian if and only if A and B commute.

3. Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that there is a vector $v \in V$ such that the list

$$\{v, Tv, T^2v, \dots\}$$

spans V. Show that if $S : V \to V$ is a linear transformation that commutes with T, then there is a polynomial f such that $S = f(T)$.

4. Consider the $n \times n$ matrix $A_n = [a_{ij}]$, where $a_{ii} = 1$ and, for all $i \neq j$ and some $-1 < \rho < 1$, $a_{ij} = \rho$. For example,

$$A_2 = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}.$$

Compute $\det(A_n)$ as a function of n.

5. Find a real matrix A such that $e^A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ or prove no such matrix can exist.