Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Monday, August 19, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let $u, v \in \mathbb{R}^n$ be arbitrary points in n-dimensional Euclidean space. Prove:

$$\Big||u| - |v|\Big| \leq |u - v|.$$

Solution: By the triangle inequality, $|u| = |v + (u - v)| \leq |v| + |u - v|$, which gives $|u-v| \geq |u|-|v|$. Similarly, $|v| = |u+(v-u)| \leq |u|+|u-v|$ gives $|u-v| \geq -(|u|-|v|)$. Together, we have $|u - v| \geq \Big||u| - |v|\Big|$.

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2. Compute the limit (with justification):

$$\lim_{x\to+\infty} \left(\sqrt{(x+5)(x+7)} - x\right).$$

Solution: First

$$\sqrt{(x+5)(x+7)} - x = \frac{(x+5)(x+7) - x^2}{\sqrt{(x+5)(x+7)} + x} = \frac{12x + 35}{\sqrt{x^2 + 12x + 35} + x},$$

from which the following bounds are immediate:

$$\frac{12x+35}{2x+6} \leq \frac{12x+35}{\sqrt{(x+6)^2} + x} \leq \sqrt{(x+5)(x+7)} - x \leq \frac{12x+35}{2x}.$$

By the squeeze theorem, the limit is 6.

Alternative solution: When $0 < (12/x) + (35/x^2) < 1$, the binomial series yields:

$$\begin{aligned}
\sqrt{(x+5)(x+7)} - x &= x\left[\left(1 + \frac{12}{x} + \frac{35}{x^2}\right)^{1/2} - 1\right] \\
&= x\left[\frac{1}{2}\left(\frac{12}{x} + \frac{35}{x^2}\right) + O\left(\left(\frac{12}{x} + \frac{35}{x^2}\right)^2\right)\right] \\
&= x\left[\frac{6}{x} + O\left(\frac{1}{x^2}\right)\right] = 6 + O\left(\frac{1}{x}\right).
\end{aligned}$$

Thus, the limit is 6.

3. Does the improper Riemann integral $\int_0^\infty \cos(x^2)\,dx$ converge or diverge? Justify your assertion. If needed, you may use (without proof) the fact that $|\frac{\sin(u)}{u}| \le 1$ for $u \neq 0$.

Solution: It converges. Here's why. An integration by parts gives

$$\int_a^b \cos(x^2)\,dx = \int_a^b \frac{1}{2x}\cdot 2x\,\cos(x^2)\,dx = \left.\frac{\sin(x^2)}{2x}\right|_{x=a}^{x=b} + \frac{1}{2}\int_a^b \frac{\sin(x^2)}{x^2}\,dx.$$

Since $0 \le |\sin(x^2)| \le 1$, $\frac{\sin(b^2)}{2b}$ will tend to 0 as b tends to $+\infty$; also, $\frac{\sin(a^2)}{2a} = \frac{a}{2}\frac{\sin(a^2)}{a^2}$ will tend to 0 as a tends to 0 since $\frac{\sin(a^2)}{a^2}$ remains bounded. Together these show the boundary term in the display above vanishes in the limit as $b \to +\infty$ and $a \to 0$. Lastly, $\int_0^\infty \frac{\sin(x^2)}{x^2}\,dx = \int_0^1 \frac{\sin(x^2)}{x^2}\,dx + \int_1^\infty \frac{\sin(x^2)}{x^2}\,dx$. The first integral here is bounded in absolute value by 1. The second integral is bounded in absolute value by $\int_1^\infty \frac{1}{x^2}\,dx = 1$.

Remark about above solution: One could have noticed that the integral in the problem statement is only improper at the upper limit $+\infty$. Consequently, convergence of the integral would follow from convergence of the integral $\int_1^\infty \cos(x^2)\,dx$ circumventing the need of using $|\sin(x)/x|$ remaining bounded near the origin.

Alternative solution: we can make a change of variable to give

$$\int_{x=0}^\infty \cos(x^2)\,dx = \frac{1}{2}\int_{u=0}^\infty \frac{\cos(u)}{\sqrt{u}}\,du = \frac{1}{2}\int_{u=0}^{\pi/2} \frac{\cos(u)}{\sqrt{u}}\,du + \frac{1}{2}\int_{u=\pi/2}^\infty \frac{\cos(u)}{\sqrt{u}}\,du.$$

The first integral is finite since the integrand is nonnegative in $(0, \pi/2]$ and bounded by $1/\sqrt{u}$, which is integrable over the interval. The second integral can be rewritten as

$$\sum_{k=0}^\infty \{A_k + B_k\}, \tag{1}$$

where

$$A_k = \int_{u=2\pi k+(\pi/2)}^{2\pi k+(3\pi/2)} \frac{\cos(u)}{\sqrt{u}}\,du,$$

and

$$B_k = \int_{u=2\pi k+(3\pi/2)}^{2\pi k+(5\pi/2)} \frac{\cos(u)}{\sqrt{u}}\,du.$$

Observe that the integral A_k is negative and the integral B_k is positive, so that the series $A_0 + B_0 + A_1 + B_1 + \cdots$ is alternating. Since $|\cos u| \le 1$, we have

$$|A_k| \le \int_{u=2\pi k+(\pi/2)}^{2\pi k+(3\pi/2)} \frac{1}{\sqrt{u}}\,du \;=\; 2\left(\sqrt{2\pi k+(3\pi/2)} - \sqrt{2\pi k+(\pi/2)}\right) \;\le\; \frac{C}{\sqrt{k+(1/4)}}, \qquad k \ge 0$$

for a constant C. Also, since $\cos(u+\pi) = -\cos(u)$ we have

$$B_k = \int_{u=2\pi k+(3\pi/2)}^{2\pi k+(5\pi/2)} \frac{\cos(u)}{\sqrt{u}}\,du \;=\; \int_{u=2\pi k+(\pi/2)}^{2\pi k+(3\pi/2)} \frac{\cos(u+\pi)}{\sqrt{u+\pi}}\,du \;=\; \left|\int_{u=2\pi k+(\pi/2)}^{2\pi k+(3\pi/2)} \frac{\cos(u)}{\sqrt{u+\pi}}\,du\right| \le |A_k|.$$

and by an identical argument $|A_{k+1}| \le B_k$. Thus, the absolute value of the terms of the series $A_0 + B_0 + A_1 + B_1 + \cdots$ tend to zero monotonically and (1) converges by Leibniz's alternating series test.

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4. If (a_n) $(n \ge 1)$ is a Cauchy sequence of real numbers, is it true that (a_n) is bounded? Prove this or give a counterexample.

Solution: The sequence must be bounded. Since (a_n) is Cauchy, for any $\varepsilon > 0$, there exists an integer N such that $|a_n - a_m| \le \varepsilon$ for all $n, m \ge N$. Since $|a_n| - |a_m| \le |a_n - a_m|$, we have (by taking $m = N$) that $|a_n| \le |a_N| + \varepsilon$ for all $n \ge N$. Thus, for all $n \ge 1$ we have $|a_n| \le \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + \varepsilon\}$.

Alternative solution: Since the real line is complete, the Cauchy sequence is convergent and, therefore, bounded.

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5. Assume that f is twice differentiable on the finite open interval (a, b), and there is $M \ge 0$ such that $|f''(x)| \le M$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b).

Solution: We first show that f' is bounded. Let z be fixed in (a, b). Let x be any point in (a, b); by the mean-value theorem, there exists some point $y \in (a, b)$ such that

$$f'(x) = f'(z) + f''(y)(x - z).$$

Hence, we have

$$|f^{'}(x)| \leq |f^{'}(z)| + M(b-a) := c.$$

By the mean-value theorem again, for any two points x and y in (a, b), we have $|f(x)-f(y)| \leq c|x-y|$. Then uniform continuity follows from this Lipschitz continuity.

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INTRODUCTORY EXAMINATION–SUMMER SESSION
AFTERNOON EXAM–PROBABILITY

Monday, August 19, 2019

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2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. A standard six-sided die is rolled twelve times. Compute the probability that one number occurs six times and two other numbers occur three times each.

Solution: The number of outcomes is 6^{12}. To count the number of good outcomes:
1. Pick the number that occurs 6 times: $\binom{6}{1} = 6$ choices.
2. Pick the two numbers that occur 3 times each: $\binom{5}{2}$ choices.
3. Pick slots (rolls) for the number that occurs 6 times: $\binom{12}{6}$ choices.
4. Pick slots for the smaller of the numbers that occur 3 times each: $\binom{6}{3}$ choices.

Therefore, our probability is $\binom{6}{1}\binom{5}{2}\binom{12}{6}\binom{6}{3}/6^{12} = \frac{1925}{3779136} \doteq 0.0051$.

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2. Seven girls and three boys are to be seated in a row of ten chairs randomly. If we think of the ten chairs as nine adjacent pairs of chairs, what is the expected number of adjacent pairs of chairs occupied by two girls?

Solution: For $i = 1, 2, \ldots, 9$ let $X_i = 1$ if the ith adjacent pair of chairs is occupied by two girls, otherwise $X_i = 0$. Then $\sum_{i=1}^{9} X_i$ is the total number of adjacent pairs of chairs occupied by two girls. It follows that $E(\sum_{i=1}^{9} X_i) = \sum_{i=1}^{9} P(X_i = 1) = 9 \cdot \frac{\binom{7}{2}}{\binom{10}{2}} = \frac{21}{5} = 4.2$.

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3. Let $\alpha > 0$ and $\beta > 0$ be arbitrary. Suppose X and Y have the joint pdf

$$f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(y-x)^{\beta-1}e^{-y} \text{ for } 0 < x < y < \infty,$$

and $f(x, y) = 0$ otherwise. Find the pdf of $U = Y - X$. Is U independent of X?

Solution: Consider the transformation $u = y - x$ and $v = x$ with inverse transformation $x = v$ and $y = u + v$ having Jacobian determinant $J = \det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = -1$.

It follows that for $0 < x < y < \infty$ we have $0 < v < u + v < \infty$ and consequently, for $u > 0$ and $v > 0$, we have

$$f_{U,V}(u,v) = f(v, u+v)|J| = \frac{v^{\alpha-1}u^{\beta-1}e^{-(u+v)}}{\Gamma(\alpha)\Gamma(\beta)} = \frac{v^{\alpha-1}e^{-v}}{\Gamma(\alpha)} \cdot \frac{u^{\beta-1}e^{-u}}{\Gamma(\beta)}.$$

This shows that U and $V = X$ are independent, as well as that $U \sim \text{Gamma}(\beta, 1)$.

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4. You toss a balanced coin until you see a head for the first time, say on trial N. Then you toss the same coin N times. Compute the probability you only toss that one head. (Here N is random.)

Solution: We are told $P(N = n) = (\frac{1}{2})^n$ for $n = 1, 2, 3, \ldots$, and let X_1, X_2, X_3, be an iid sequence of Bernoulli($\frac{1}{2}$) random variables independent of N. We wish to compute $P(\sum_{i=1}^N X_i = 0)$. Conditionally given $N = n$, $\sum_{i=1}^N X_i$ is distributed binomial($n, \frac{1}{2}$) and the probability we see no heads in these n flips is $(\frac{1}{2})^n$. Consequently, by the law of total probability $P(\sum_{i=1}^N X_i = 0) = \sum_{n=1}^\infty (\frac{1}{2})^{2n} = \frac{1}{3}$.

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5. Let X and Y be jointly continuous random variables having the joint pdf $f(x, y) = e^{-y}$ for $0 < x < y < \infty$ and $f(x, y) = 0$ otherwise. Compute $\text{Var}(X)$ and $\text{Var}(X|Y)$.

Solution: Compute the densities for X and Y and the conditional density for X given Y:

$$f_X(x) = e^{-x} \quad \text{for } x > 0,$$
$$f_Y(y) = ye^{-y} \quad \text{for } y > 0, \text{ and}$$
$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{y} \quad \text{for } 0 < x < y.$$

Notice that X is an exponential random variable with mean 1, and that the conditional distribution of X given $Y = y$ is uniform on $(0, y)$.

Therefore, conditionally, the distribution of X given $Y = y$ is that of yU where $U \sim \text{Uniform}(0, 1)$; hence the conditional variance of X given $Y = y$ is $y^2/12$. Also, since X is an exponential random variable with mean 1, we know

$$\text{Var}(X) = 1.$$

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Morning Exam–Linear Algebra

Tuesday, August 20, 2019

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1. Let A be an $n \times n$ skew symmetric matrix. Prove that A is singular when n is odd.

Solution: Since $A = -A^T$, we have $\det(A) = (-1)^n \det(A^T) = -\det(A)$ implies $\det(A) = 0$.

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2. Suppose A and B are two positive definite $n \times n$ real matrices. Prove that $A^{-1} + B^{-1}$ exists and is invertible.

Solution: Since each of A and B are positive definite, they have non-zero determinant and are therefore each invertible. Moreover, A^{-1} and B^{-1} are also positive definite, and the sum of positive definite matrices is positive definite.

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3. Let A and B be any two $n \times n$ real matrices that share a common eigenvector. Prove: $\det(AB - BA) = 0$.

Solution: Let v be the common eigenvector so that there exist complex numbers a and b such that $Av = av$ and $Bv = bv$. Since $v \neq 0$ and $(AB - BA)v = A(bv) - B(av) = bav - abv = 0$, we've shown the columns of $AB - BA$ are linearly dependent and therefore the matrix $AB - BA$ is singular.

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4. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T have only the zero vector in common.

Solution: Let $\{v_1, \ldots, v_n\}$ be a basis for V. Then the rank of T is the maximum number of linearly independent vectors in the set $\{Tv_1, \ldots, Tv_n\}$. Suppose the rank of T equals k and suppose without loss of generality that $\{Tv_1, \ldots, Tv_k\}$ is a linearly independent set (it might be that $k = 1$). Then $\{Tv_1, \ldots, Tv_k\}$ is a basis for the range of T. It follows that $\{T^2v_1, \ldots, T^2v_k\}$ spans the range of T^2, and since the dimension of the range of T^2 is also equal to k, $\{T^2v_1, \ldots, T^2v_k\}$ must be a basis for the range of T^2. Now suppose v is in the range of T. Then $v = c_1Tv_1 + \cdots + c_kTv_k$. Suppose v is also in the null space of T. Then, for some scalars $c_1, \ldots, c_k$, $0 = T(v) = T(c_1Tv_1 + \cdots + c_kTv_k) = c_1T^2v_1 + \cdots + c_kT^2v_k$. But $\{T^2v_1, \ldots, T^2v_k\}$ is a basis, so $T^2v_1, \ldots, T^2v_k$ are linearly independent; thus it must be that $c_1 = \cdots = c_k = 0$, which implies $v = 0$. Thus we have shown that if v is in both the range of T and the null space of T then $v = 0$, as required.

Alternative solution:

T is a linear mapping from V onto $W := \mathrm{im}(T)$ and the restriction $T|_W$ of T to W is a linear mapping onto $U := \mathrm{im}(T^2)$. By the rank assumption, $\dim(W) = \dim(U)$ so the dimension of the image of $T|_W$ is the same as the dimension of its domain. It follows that the dimension of the null space of $T|_W$ is zero. Thus, if $w = Tv$ for some $v \in V$ and $Tw = 0$ then $w \in W$ and $T|_W(w) = 0$. We conclude therefore that $w = 0$.

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5. Decompose the following matrix into the sum of two rank-1 matrices:

$$\begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 1 & 2 & -3 & 7 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix is easily seen to be rank 2 as the following row echelon form shows:

$$\begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two row vectors in this echelon form also form a basis for the row space, which gives us a good starting point:

$$\begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 1 & 2 & -3 & 7 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 & -7 \end{bmatrix}.$$

Alternative solution: Letting the 1st, 2nd, and 3rd rows of the matrix be denoted by u, v, w, we have $w = v - u$ so the matrix can be written

$$\begin{bmatrix} u \\ v \\ v - u \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ 0 \\ -u \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} v.$$

Since u and v are non-zero, the row-ranks of the two matrices here are both 1.
