Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Monday, August 19, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let $u, v \in \mathbb{R}^n$ be arbitrary points in n-dimensional Euclidean space. Prove:

$$\Big| |u| - |v| \Big| \leq |u - v|.$$

2. Compute the limit (with justification):

$$\lim_{x \to +\infty} \left(\sqrt{(x+5)(x+7)} - x \right).$$

3. Does the improper Riemann integral $\int_0^\infty \cos(x^2)\, dx$ converge or diverge? Justify your assertion. If needed, you may use (without proof) the fact that $|\frac{\sin(u)}{u}| \leq 1$ for $u \neq 0$.

4. If (a_n) $(n \geq 1)$ is a Cauchy sequence of real numbers, is it true that (a_n) is bounded? Prove this or give a counterexample.

5. Assume that f is twice differentiable on the finite open interval (a, b), and there is $M \geq 0$ such that $|f''(x)| \leq M$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b).

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Introductory Examination–Summer Session
Afternoon Exam–Probability

Monday, August 19, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A standard six-sided die is rolled twelve times. Compute the probability that one number occurs six times and two other numbers occur three times each.

2. Seven girls and three boys are to be seated in a row of ten chairs randomly. If we think of the ten chairs as nine adjacent pairs of chairs, what is the expected number of adjacent pairs of chairs occupied by two girls?

3. Let $\alpha > 0$ and $\beta > 0$ be arbitrary. Suppose X and Y have the joint pdf

$$f(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(y-x)^{\beta-1}e^{-y} \text{ for } 0 < x < y < \infty,$$

and $f(x, y) = 0$ otherwise. Find the pdf of $U = Y - X$. Is U independent of X?

4. You toss a balanced coin until you see a head for the first time, say on trial N. Then you toss the same coin N times. Compute the probability you only toss that one head. (Here N is random.)

5. Let X and Y be jointly continuous random variables having the joint pdf $f(x, y) = e^{-y}$ for $0 < x < y < \infty$ and $f(x, y) = 0$ otherwise. Compute $\mathrm{Var}(X)$ and $\mathrm{Var}(X|Y)$.

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Morning Exam–Linear Algebra

Tuesday, August 20, 2019

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let A be an $n \times n$ skew symmetric matrix. Prove that A is singular when n is odd.

2. Suppose A and B are two positive definite $n \times n$ real matrices. Prove that $A^{-1} + B^{-1}$ exists and is invertible.

3. Let A and B be any two $n \times n$ real matrices that share a common eigenvector. Prove: $\det(AB - BA) = 0$.

4. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T have only the zero vector in common.

5. Decompose the following matrix into the sum of two rank-1 matrices:

$$\begin{bmatrix} 1 & 2 & -3 & 7 & 0 & -2 & 5 \\ 1 & 2 & -3 & 7 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$