Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Monday, August 20, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose f is continuous on the closed interval $[0,2]$ with $f(0) = f(2)$. Show that there exists points $x_1, x_2 \in [0,2]$ with $x_2 = x_1 + 1$ such that $f(x_1) = f(x_2)$.

Solution: Let $g(x) = f(x) - f(x+1)$ so that g is continuous on the closed interval $[0,1]$. We have

$$g(0) = f(0) - f(1) = -[f(1) - f(2)] = -g(1).$$

Therefore, by the Intermediate Value Theorem, there exists an $x_1 \in [0,1]$ such that $g(x_1) = 0$. Let $x_2 = x_1 + 1$. Then $g(x_1) = 0$ implies $f(x_1) = f(x_2)$.

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2. Let $a_0 = 1$ and, for nonnegative integers k, let $a_{k+1} = \sqrt{a_k + 1}$.

Show that

$$\lim_{k\to\infty} a_k$$

exists and find its value.

Solution: We claim

$$\lim_{k\to\infty} a_k = \frac{1+\sqrt{5}}{2}.$$

Let $f(x) = \sqrt{1+x}$. Note that

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

which implies that $0 < f'(x) \le \frac{1}{2}$ for all $x \ge 0$. For distinct $x, y \ge 0$ we have, by the Mean Value Theorem,

$$\frac{f(x) - f(y)}{x - y} = f'(z) \in (0, \tfrac{1}{2}]$$

for some z in the interval between x and y. This implies that $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ and therefore f is a contraction mapping on the nonnegative reals. It follows therefore that the sequence converges to the unique fixed point of f which is $(1+\sqrt{5})/2$.

Alternate solution. Let $b := (1+\sqrt{5})/2$. It is immediate by induction that $a_k \in [1, b)$ for $k \ge 0$. We will prove by induction that $a_{k+1} - a_k > 0$ for $k \ge 0$, thus establishing that the sequence (a_k) is strictly increasing and therefore has a limit $\lambda \in [1, b]$. Passing to the limit in the defining equation for a_{k+1}, we then find $\lambda = \sqrt{\lambda + 1}$, from which it follows easily that $\lambda = b$. Now we prove that $a_{k+1} - a_k > 0$ for $k \ge 0$. The basis of the induction is the observation that $a_1 - a_0 = \sqrt{2} - 1 > 0$. For the induction step, let $k \ge 1$. Then $a_{k+1} - a_k = \sqrt{a_k + 1} - \sqrt{a_{k-1} + 1} > 0$ by the induction hypothesis because $\sqrt{\cdot + 1}$ is strictly increasing.

3. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ with n even, and real coefficients $a_0, \ldots, a_n$ with $a_n > 0$. Show that f has a global minimum.

Solution: The function f is continuous, so if f were not to have a global minimum it would have to be the case that there is a sequence (x_k) with $|x_k| \to \infty$ such that $f(x_k) \to -\infty$. But the hypotheses clearly imply that $f(x) \to +\infty$ as $|x| \to \infty$.

Alternate solution. We can, for every $M > f(0)$, find a closed interval $[-N, N]$ such that $f(x) \geq M$ for all $x \notin [-N, N]$. Moreover, for any such M, since the function f is continuous on $[-N, N]$, it obtains its minimum value, say at the point $x_* \in [-N, N]$, i.e., $f(x_*) \leq f(0) < M$. This implies that x_* is a global minimizer of f.

4. Consider the following power series $L(x)$, which is also known as *Euler's dilogarithm function*:

$$L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

Compute the domain of convergence for $L(x)$ and show that $L(x)$ is continuous on its domain.

Solution: We use the ratio test to get the radius of convergence:

$$\lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} \right| = |x|.$$

This shows that the radius of convergence of the series is 1. When $x = 1$ the series is a convergent p-series (with $p = 2$) and when $x = -1$ it is a convergent alternating series. So the domain of convergence is $[-1, 1]$.

Since $\left|\frac{x^k}{k^2}\right| \leq \frac{1}{k^2}$ for $x \in [-1, 1]$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, the Weierstrass M-test tells us that the power series $L(x)$ converges uniformly on its domain of convergence $[-1, 1]$. Now, since the partial sums are each continuous and are uniformly converging, the limit function is also continuous.

5. Suppose that $0 \le a_{i,j} < \infty$ for all integers $i, j = 1, 2, 3, \ldots$. Prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}.$$

Prove this from first principles; do not, for instance, appeal to the Fubini–Tonelli theorem.

Solution: Set $A = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}$ and $B = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$, where we allow for the possibility these may be infinite. Now, for any integers $m \ge 1$ and $n \ge 1$ we have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i,j} \le \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{i,j} \le \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j} = B,$$

from which it follows that $A \le B$. A completely symmetric argument shows $B \le A$, and the result follows.

Department of Applied Mathematics and Statistics
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INTRODUCTORY EXAMINATION–SUMMER SESSION
AFTERNOON EXAM–PROBABILITY

Monday, August 20, 2018

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5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Prove for any integer $n \geq 2$ and events $A_1, A_2, \ldots, A_n$ that

$$P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum\sum_{1\leq i<j\leq n} P(A_i \cap A_j).$$

Solution: Here's a proof by induction. The statement is true when $n = 2$ since this is just a restatement of the inclusion exclusion principle:

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Now assume the statement is true for some $n \geq 2$. Write $\bigcup_{i=1}^{n+1} A_i = A_{n+1} \cup \bigcup_{i=1}^{n} A_i$, and use the inclusion exclusion principle and Boole's inequality to obtain

$$\begin{aligned} P(\bigcup_{i=1}^{n+1} A_i) &= P(A_{n+1}) + P(\bigcup_{i=1}^{n} A_i) - P(A_{n+1} \cap \bigcup_{i=1}^{n} A_i) \\ &\geq P(A_{n+1}) + \sum_{i=1}^{n} P(A_i) - \sum_{j=2}^{n}\sum_{i=1}^{j-1} P(A_i \cap A_j) - \sum_{i=1}^{n} P(A_i \cap A_{n+1}) \\ &= \sum_{i=1}^{n+1} P(A_i) - \sum_{j=2}^{n+1}\sum_{i=1}^{j-1} P(A_i \cap A_j) \end{aligned}$$

which completes the induction.

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2. An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks (one female, one male) for breeding. Assume yaks are loners and roam about the Himalayas at random. We denote by p the probability that a given trapped yak is male, and we assume the event that a trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a (male/female) pair is obtained for the first time. Compute the mean of N.

Solution: The discrete random variable N takes values $2, 3, 4, \ldots$ since we will need to trap at least two yaks to get a male/female pair. The event $(N = 2) = \{(F, M), (M, F)\}$, where, for instance, (F, M) denotes the outcome that the first yak trapped is female and the second is male. For $k > 2$, the event $(N = k) = \{(\underbrace{F, \ldots, F}_{k-1}, M), (\underbrace{M, \ldots, M}_{k-1}, F)\}$.

Therefore, if we let $X \sim \text{geometric}(p)$ and $Y \sim \text{geometric}(1-p)$, then for any $k \geq 2$, $P(N = k) = P(X = k) + P(Y = k)$. It follows that

$$\begin{aligned} E(N) &= \sum_{k=2}^{\infty} kP(N = k) = \sum_{k=2}^{\infty} kP(X = k) + \sum_{k=2}^{\infty} kP(Y = k) \\ &= E(X) - P(X = 1) + E(Y) - P(Y = 1) = \frac{1}{p} - p + \frac{1}{1-p} - (1-p) \\ &= \frac{1}{p} + \frac{1}{1-p} - 1 \quad = \quad \frac{1}{p(1-p)} - 1. \end{aligned}$$

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3. Let X be a unit exponential random variable, i.e., having pdf $f(x) = e^{-x}$ for $x > 0$, and let $c > 0$ be a fixed constant. Set $Y = \max\{X, c\}$. Compute $E(X|Y = y)$ for values of $y \geq c$.

Solution:

If $y > c$, then $Y = y$ is equivalent to $X = y$ and consequently $E(X|Y = y) = y$.

If $y = c$, then $Y = y$ is equivalent to $0 < X \leq c$. In this case

$$E(X|0 < X \leq c) = \int_0^c x \frac{e^{-x}}{1 - e^{-c}}\, dx = \frac{1 - e^{-c} - ce^{-c}}{1 - e^{-c}} = 1 - \frac{ce^{-c}}{1 - e^{-c}}.$$

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4. Suppose X is a discrete random variable taking positive integer values $1, 2, 3, \ldots$ and the pmf of X is nonincreasing. Show that for any integer $k \geq 1$,

$$P(X = k) \leq \frac{2E(X)}{k(k+1)}.$$

Solution:

$$\begin{aligned} E(X) &= \sum_{i=1}^{\infty} iP(X=i) \\ &\geq \sum_{i=1}^{k} iP(X=i) \\ &\geq \sum_{i=1}^{k} iP(X=k) \\ &= P(X=k)\sum_{i=1}^{k} i = \frac{k(k+1)}{2}P(X=k). \end{aligned}$$

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5. A four digit number is selected at random. What is the probability that its leading digit is strictly larger than its second digit, its second digit is strictly larger than its third digit, and its third digit is strictly larger than its fourth digit? [Note that the leading digit of an n-digit number is nonzero. For example, there are ninety 2-digit numbers: ten with leading digit 1, ..., ten with leading digit 9.]

Solution: Use an equally-likely model, in which each of the possible 4-digit numbers are equally likely.

For the denominator, the leading digit can be $1, 2, 3, \ldots, 9$, but not 0, while each of the other digits can be $0, 1, 2, 3, \ldots, 9$, there are $9(10)^3 = 9000$ possible 4-digit numbers.

For the numerator, the 4 digits must be different, and must be in decreasing order. Thus, each four digit number in the event is obtained by choosing a subset of 4 of the 10 possible digits and placing them in decreasing order.

The desired probability is then

$$\frac{\binom{10}{4}}{9000} = \frac{210}{9000} = \frac{7}{300} = .023333\ldots.$$

Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Linear Algebra

Tuesday, August 21, 2018

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7. **No calculators of any sort are needed or permitted.**

1. Let A be a real $n \times n$ matrix with $\|A\| < 1$. Here, you may assume $\|\cdot\|$ is any matrix norm induced by a norm on $\mathbb{R}^n$. Prove that $I - A$ is nonsingular, and

$$\|(I - A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

Solution: Let $x \in \mathbb{R}^n$, $x \neq 0$. Then

$$\|(I - A)x\| = \|x - Ax\| \geq \|x\| - \|Ax\| \geq \|x\| - \|A\|\|x\| = (1 - \|A\|)\|x\| > 0,$$

since $1 - \|A\| > 0$. Thus, if $x \neq 0$, then $(I - A)x \neq 0$, and $I - A$ is nonsingular. Now from the equation

$$(I - A)(I - A)^{-1} = I,$$

it follows that

$$(I - A)^{-1} = I + A(I - A)^{-1}.$$

Thus,

$$\|(I - A)^{-1}\| \leq \|I\| + \|A\|\|(I - A)^{-1}\|,$$

and since $\|I\| = 1$, the result follows.

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2. For a nonnegative integer k, let $x^{\underline{k}}$ be the polynomial

$$x^{\underline{k}} = \underbrace{(x)(x-1)(x-2)\cdots(x-k+1)}_{k \text{ factors}}.$$

Note that $x^{\underline{0}} = 1$.

Let $f(x)$ be an arbitrary polynomial (say, with real coefficients). Prove that we can express $f(x)$ *uniquely* as a finite linear combination of $x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \ldots$.

Solution: Let d be the degree of $f(x)$. Let $\mathcal{V}_d$ be the vector space of all polynomials of degree at most d. This is a $d + 1$ dimensional vector space with natural basis $\{1, x, x^2, \ldots, x^d\}$.

Claim: $x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \ldots, x^{\underline{d}}$, which are all elements of $\mathcal{V}_d$, are linearly independent.

Suppose, for contradiction, $x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \ldots, x^{\underline{d}}$ are not linearly independent and choose coefficients $a_0, \ldots, a_d$, not all zero, so that

$$a_0 x^{\underline{0}} + a_1 x^{\underline{1}} + \cdots + a_d x^{\underline{d}} = 0.$$

Since the coefficient are not all zero, let k be the largest index so that $a_k \neq 0$. Then we have

$$\underline{x}^{k} = -\frac{a_0}{a_k}\underline{x}^{0} - \frac{a_1}{a_k}\underline{x}^{1} - \cdots - \frac{a_{k-1}}{a_k}\underline{x}^{k-1}. \qquad (*)$$

However, the LHS of $(*)$ is a polynomial of degree k and the RHS is a polynomial of degree (at most) $k-1$, and that's a contradiction.

It now follows that $\{\underline{x}^{0}, \underline{x}^{1}, \underline{x}^{2}, \ldots, \underline{x}^{d}\}$ form a basis for $\mathcal{V}_d$, and therefore $f \in \mathcal{V}_d$ can be written uniquely as a linear combination of these basis elements. Furthermore, it is clear that such f cannot be written as a finite linear combination of the polynomials $\underline{x}^{k}$ with any nonzero coefficients for $k > d$.

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3. Suppose A and B are 2×2 matrices. Show that $(AB - BA)^2$ commutes with every 2×2 matrix, that is, $(AB - BA)^2 C = C(AB - BA)^2$ for every 2×2 matrix C.

Hint: What is the trace of $AB - BA$?

Solution: We claim that $(AB - BA)^2$ is a multiple of the identity matrix. Following the hint, first observe that

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

So we can write

$$AB - BA = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

for some choice of constants x, y and z. Now we calculate

$$(AB - BA)^2 = \begin{bmatrix} x^2 + yz & 0 \\ 0 & x^2 + yz \end{bmatrix} = (x^2 + yz)I_2.$$

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4. Let $A = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$ with $B \in \mathbb{C}^{k \times k}$ for some $1 \leq k < n$. Prove that A is normal if and only if B is normal and $C = 0$.

Recall: If A is an $n \times n$ matrix, then A^* is its conjugate transpose and A is normal provided $A^*A = AA^*$.

Solution: We have

$$A^*A = \begin{bmatrix} B^*B & B^*C \\ C^*B & C^*C \end{bmatrix} \quad \text{and} \quad AA^* = \begin{bmatrix} BB^* + CC^* & 0 \\ 0 & 0 \end{bmatrix}. \qquad (1)$$

Also note that $C \in \mathbb{C}^{k\times(n-k)}$. We now prove the two implications in turn.

It follows immediately from (1) that if B is normal and $C = 0$, then $A^*A = AA^*$ so that A is normal. This proves the "if" direction.

To prove the "only if" direction, suppose that A is normal so that $A^*A = AA^*$. It follows from this fact and (1) that

$$\begin{bmatrix} B^*B & B^*C \\ C^*B & C^*C \end{bmatrix} = \begin{bmatrix} BB^* + CC^* & 0 \\ 0 & 0 \end{bmatrix}. \tag{2}$$

This implies, in particular, that

$$C^*C = 0.$$

Using this fact, we may then write

$$\|Cx\|_2^2 = (Cx)^*(Cx) = x^*C^*Cx = x^*(C^*C)x = 0 \quad \text{for all} \quad x \in \mathbb{C}^n.$$

Therefore $Cx = 0$ for all $x \in \mathbb{C}^n$ and so $C = 0$, as desired. Combining this with the conditions in the (1,1) block of (2) shows that $B^*B = BB^*$, so that B is normal. This completes the proof.

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5. For vectors $x \in \mathbb{R}^n$, let $\|x\| = \max_{1\le i\le n} |x_i|$ denote the sup-norm, and for $n \times n$ (real) matrices $A = (a_{ij})$, let $\|A\| = \sup\{\|Au\| : u \in \mathbb{R}^n, \|u\| = 1\}$. Show that $\|A\| = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$.

Solution:

$$\begin{aligned}
\|A\| &= \sup_{\|u\|=1} \|Au\| \\
&= \sup_{\|u\|=1} \max_{1\le i\le n} |(Au)_i| \\
&= \max_{1\le i\le n} \sup_{\|u\|=1} |(Au)_i| \\
&= \max_{1\le i\le n} \sup_{\|u\|=1} \left|\sum_{j=1}^n a_{ij}u_j\right| \\
&= \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.
\end{aligned}$$

Here we used the fact that the supremum of $|\sum_{j=1}^n a_{ij}u_j|$ for fixed i and $\|u\| = 1$ is obtained by putting $u_j = +1$ if $a_{ij} \ge 0$, and $u_j = -1$ if $a_{ij} < 0$.
