Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Monday, August 20, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Suppose f is continuous on the closed interval $[0, 2]$ with $f(0) = f(2)$. Show that there exists points $x_1, x_2 \in [0, 2]$ with $x_2 = x_1 + 1$ such that $f(x_1) = f(x_2)$.

2. Let $a_0 = 1$ and, for nonnegative integers k, let $a_{k+1} = \sqrt{a_k + 1}$.

   Show that

$$\lim_{k\to\infty} a_k$$

   exists and find its value.

3. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ with n even, and real coefficients $a_0, \ldots, a_n$ with $a_n > 0$. Show that f has a global minimum.

4. Consider the following power series $L(x)$, which is also known as *Euler's dilogarithm function*:

$$L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

   Compute the domain of convergence for $L(x)$ and show that $L(x)$ is continuous on its domain.

5. Suppose that $0 \le a_{i,j} < \infty$ for all integers $i, j = 1, 2, 3, \ldots$. Prove that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}.$$

   Prove this from first principles; do not, for instance, appeal to the Fubini–Tonelli theorem.

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Afternoon Exam–Probability

Monday, August 20, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps.** The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Prove for any integer $n \geq 2$ and events $A_1, A_2, \ldots, A_n$ that

$$P(\bigcup_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \leq i < j \leq n} P(A_i \cap A_j).$$

2. An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks (one female, one male) for breeding. Assume yaks are loners and roam about the Himalayas at random. We denote by p the probability that a given trapped yak is male, and we assume the event that a trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a (male/female) pair is obtained for the first time. Compute the mean of N.

3. Let X be a unit exponential random variable, i.e., having pdf $f(x) = e^{-x}$ for $x > 0$, and let $c > 0$ be a fixed constant. Set $Y = \max\{X, c\}$. Compute $E(X|Y = y)$ for values of $y \geq c$.

4. Suppose X is a discrete random variable taking positive integer values $1, 2, 3, \ldots$ and the pmf of X is nonincreasing. Show that for any integer $k \geq 1$,

$$P(X = k) \leq \frac{2E(X)}{k(k+1)}.$$

5. A four digit number is selected at random. What is the probability that its leading digit is strictly larger than its second digit, its second digit is strictly larger than its third digit, and its third digit is strictly larger than its fourth digit? [Note that the leading digit of an n-digit number is nonzero. For example, there are ninety 2-digit numbers: ten with leading digit 1, ..., ten with leading digit 9.]

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
MORNING EXAM–LINEAR ALGEBRA

Tuesday, August 21, 2018

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let A be a real $n \times n$ matrix with $\|A\| < 1$. Here, you may assume $\|\cdot\|$ is any matrix norm induced by a norm on $\mathbb{R}^n$. Prove that $I - A$ is nonsingular, and

$$\|(I-A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

2. For a nonnegative integer k, let $x^{\underline{k}}$ be the polynomial

$$x^{\underline{k}} = \underbrace{(x)(x-1)(x-2)\cdots(x-k+1)}_{k \text{ factors}}.$$

Note that $x^{\underline{0}} = 1$.

Let $f(x)$ be an arbitrary polynomial (say, with real coefficients). Prove that we can express $f(x)$ *uniquely* as a finite linear combination of $x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, \ldots$.

3. Suppose A and B are 2×2 matrices. Show that $(AB - BA)^2$ commutes with every 2×2 matrix, that is, $(AB - BA)^2 C = C(AB - BA)^2$ for every 2×2 matrix C.

Hint: What is the trace of $AB - BA$?

4. Let $A = \begin{bmatrix} B & C \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$ with $B \in \mathbb{C}^{k \times k}$ for some $1 \leq k < n$. Prove that A is normal if and only if B is normal and $C = 0$.

Recall: If A is an $n \times n$ matrix, then A^* is its conjugate transpose and A is normal provided $A^*A = AA^*$.

5. For vectors $x \in \mathbb{R}^n$, let $\|x\| = \max_{1 \leq i \leq n} |x_i|$ denote the sup-norm, and for $n \times n$ (real) matrices $A = (a_{ij})$, let $\|A\| = \sup\{\|Au\| : u \in \mathbb{R}^n, \|u\| = 1\}$. Show that $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$.