Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session
Morning Exam–Real Analysis

Monday, August 21, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Compute

$$\int_0^\infty \lfloor x \rfloor e^{-x}\, dx,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Simplify completely.

Solution:

$$\begin{aligned}
\int_0^\infty \lfloor x \rfloor e^{-x}\, dx &= \sum_{n=0}^{\infty} \int_n^{n+1} \lfloor x \rfloor e^{-x}\, dx \\
&= \sum_{n=0}^{\infty} n \int_n^{n+1} e^{-x}\, dx \\
&= \sum_{n=1}^{\infty} n(e^{-n} - e^{-(n+1)}) \\
&= e^{-1} + e^{-2} + e^{-3} + e^{-4} + \cdots = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1}.
\end{aligned}$$

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2. Prove that, if $f : (-a, a) \to \mathbb{R}$ is C^2 (with $a > 0$), one has

$$f(x) = f(0) + \frac{1}{2}(f'(x) + f'(0))x + o(x^2)$$

near $x = 0$.

Solution: Simply write

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + o(x^2)$$

and

$$f'(x) = f'(0) + xf''(0) + o(x)$$

to compute

$$f(x) - xf'(x)/2 = f(0) + xf'(0)/2 + o(x^2),$$

which gives the conclusion.

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3. Suppose (X, d) is a *compact* metric space. Let $f : X \to X$ be such that

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y \in X.$$

Show that there exists a unique $x^* \in X$ such that $f(x^*) = x^*$.
Hint: Consider the function $\phi(x) := d(x, f(x))$ for $x \in X$.

Solution: The function $\phi(x) = d(x, f(x))$ is seen to be continuous (this should be proved rigorously in the student's solution). Since X is compact, $\phi(x)$ has a minimizer by Weierstrauss' theorem. Let this minimizer be x^*. We claim $f(x^*) = x^*$. If not, consider $y^* = f(x^*)$. But then $\phi(y^*) = d(y^*, f(y^*)) = d(f(x^*), f(y^*)) < d(x^*, y^*) = d(x^*, f(x^*)) = \phi(x^*)$, which contradicts the fact that x^* is the minimizer for ϕ.

The uniqueness of x^* follows from the following argument. Suppose there exists $y^* \neq x^*$ such that $f(y^*) = y^*$. $d(f(x^*), f(y^*)) < d(x^*, y^*) = d(f(x^*), f(y^*))$, which is a contradiction.

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4. Calculate the Fourier coefficients $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$ of the function

$$f(x) = \begin{cases} +1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Use your result to establish the values of the following infinite series:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}, \quad \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Carefully justify all steps.

Hint: Use Parseval's theorem.

Solution: By elementary integrations

$$c_n = \frac{1 - \cos(n\pi)}{\pi i n} = \begin{cases} 2/\pi i n & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus, Parseval's theorem gives

$$\sum_{n \text{ odd}} \frac{4}{\pi^2 n^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \cdot dx = 1$$

or

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$

taking into account both positive and negative n.

Now write

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \frac{1}{4}S,$$

using absolute summability of the series to re-order it into sub-sums of even and odd terms. Solving gives

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

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5. Let m be a natural number and $f : \mathbb{R}^n \to \mathbb{R}$ be such that for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda^m f(x)$ [Such functions are called homogeneous of degree m]. Show that for all $y \in \mathbb{R}^n$, we have

$$f(y) = \frac{1}{m} \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}\bigg|_{x=y}$$

Solution: For any $y \in \mathbb{R}^n$, define $g : \mathbb{R} \to \mathbb{R}$ as $g(\lambda) = f(\lambda y)$. Since $f(\lambda y) = \lambda^m f(y)$, we obtain that $g'(\lambda) = m\lambda^{m-1} f(y)$. Also, applying the chain rule, we obtain that $g'(\lambda) = \nabla f(\lambda y)^T y$. Thus, $\nabla f(\lambda y)^T y = m\lambda^{m-1} f(y)$. Setting $\lambda = 1$ gives us the desired relation.

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SUMMER SESSION
AFTERNOON EXAM–PROBABILITY

Monday, August 21, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Assume that the random variables X and Y are jointly Gaussian, with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 1$ and $E(XY) = \rho$. Let $a \in \mathbb{R}$ be fixed. Prove that the conditional distribution of Y given $X \geq a$ has a probability density function given by

$$f(y|X \geq a) = \frac{1 - \Phi(a; \rho y, 1-\rho^2)}{1 - \Phi(a; 0, 1)} \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

where $\Phi(\cdot; m, \sigma^2)$ is the cumulative distribution function of a Gaussian variable with mean m and variance σ^2.

Solution: The joint density of X and Y is

$$\begin{aligned} g(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}(x,y) \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right) \end{aligned}$$

The conditional density is

$$f(y|X \geq a) = \frac{\int_a^\infty g(x,y)dx}{P(X \geq a)}.$$

Let's justify this quickly. For a subset $y \in \mathbb{R}$, we have

$$P(Y \leq y | X \geq a) = \frac{P(Y \leq y, X \geq a)}{P(X \geq a)} = \int_\infty^y \left(\frac{\int_a^\infty g(x,y')dx}{P(X \geq a)} \right) dy'$$

and the density is obtained by taking the derivative in y.

The denominator is by definition $1 - \Phi(a; 0, 1)$ since $X \sim \mathcal{N}(0,1)$. For the numerator, we write

$$g(x,y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

so that the integral with respect to x is

$$(1 - \Phi(a; \rho y, 1-\rho^2)) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

which is the required expression.

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2. In general, we know uncorrelated random variables are not necessarily independent. However, suppose X_1 and X_2 are Bernoulli random variable taking the values 0 and 1. Show that if X_1 and X_2 are uncorrelated then they are independent as well.

Solution: Suppose X_1 and X_2 are uncorrelated. Then $E(X_1X_2) = P(X_1 = 1, X_2 = 1)$, $E(X_1) = P(X_1 = 1)$, and $E(X_2) = P(X_2 = 1)$. Therefore, X_1 and X_2 uncorrelated implies

$$P(X_1 = 1, X_2 = 1) = P(X_1 = 1)P(X_2 = 1). \tag{1}$$

Now, using (1), $P(X_1 = 1, X_2 = 0) = P(X_1 = 1) - P(X_1 = 1, X_2 = 1) = P(X_1 = 1) - P(X_1 = 1)P(X_2 = 1) = P(X_1 = 1)[1 - P(X_2 = 1)] = P(X_1 = 1)P(X_2 = 0)$, that is,

$$P(X_1 = 1, X_2 = 0) = P(X_1 = 1)P(X_2 = 0). \tag{2}$$

In a similar way,

$$P(X_1 = 0, X_2 = 1) = P(X_1 = 0)P(X_2 = 1). \tag{3}$$

Finally, $P(X_1 = 0, X_2 = 0) = 1 - P([X_1 = 1] \cup [X_2 = 1]) = 1 - P(X_1 = 1) - P(X_2 = 1) + P(X_1 = 1, X_2 = 1) = 1 - P(X_1 = 1) - P(X_2 = 1)[1 - P(X_1)] = (1 - P(X_1 = 1))(1 - P(X_2 = 1)) = P(X_1 = 0)P(X_2 = 0)$, and combining this with (1), (2), and (3) we see X_1 and X_2 are independent as well.

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3. Distribute n balls independently and uniformly at random among n boxes. Let N_n denote the number of empty boxes. Show that for any $\varepsilon > 0$, there is a μ_n such that

$$P\Big(\Big|\frac{N_n}{n} - \mu_n\Big| > \varepsilon\Big) \to 0 \quad \text{as } n \to \infty.$$

Be sure to compute $\lim_{n\to\infty} \mu_n$.

Solution: Let $I_i = 1$ is box i is empty, $= 0$ otherwise.

Then $E(I_i) = P(I_i = 1) = (\frac{n-1}{n})^n = (1 - \frac{1}{n})^n$. Moreover, when $i \neq j$, $E(I_iI_j) = P(I_i = 1, I_j = 1) = P(\text{boxes } i \text{ and } j \text{ are empty}) = \big(\frac{n-2}{n}\big)^n = (1 - \frac{2}{n})^n$. Therefore, $var(I_i) = (1 - \frac{1}{n})^n - (1 - \frac{1}{n})^{2n}$ and for $i \neq j$, $cov(I_i, I_j) = (1 - \frac{2}{n})^n - (1 - \frac{1}{n})^{2n}$.

Now, $E(N_n) = n(1 - \frac{1}{n})^n$ so that $\mu_n = E(N_n/n) = (1 - \frac{1}{n})^n \to e^{-1}$ as $n \to \infty$.

Also, $var(N_n) = n\, var(I_1) + n(n-1)cov(I_1, I_2) = n\left((1 - \frac{1}{n})^n - (1 - \frac{1}{n})^{2n}\right) + n(n-1)\left((1 - \frac{2}{n})^n - (1 - \frac{1}{n})^{2n}\right) = n(1-\frac{1}{n})^n + n^2(1-\frac{2}{n})^n - n(1-\frac{2}{n})^n - n^2(1-\frac{1}{n})^{2n}$. Therefore,

$var(N_n/n) = \frac{1}{n^2} var(N_n) = \frac{(1-\frac{1}{n})^n}{n} + (1-\frac{2}{n})^n - \frac{(1-\frac{2}{n})^n}{n} - (1-\frac{1}{n})^{2n}$. Notice, in this last expression that as $n \to \infty$ the first and third terms go to 0 while the second term approaches e^{-2} and the last term approaches $(e^{-1})^2 = e^{-2}$ so that overall, as $n \to \infty$, $var(N_n/n) \to 0$.

Let $\varepsilon > 0$. By the Chebyshev inequality, $P\Big(\big| \frac{N_n}{n} - \mu_n \big| > \varepsilon \Big) \le \frac{var(N_n/n)}{\varepsilon^2} \to 0$ as $n \to \infty$, while $\mu_n = (1-\frac{1}{n})^n \to 1/e$ as $n \to \infty$.

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4. Let Z_1 and Z_2 be independent standard normal random variables and let $b > 0$ be a fixed constant. Find the pdf of $U = \frac{bZ_1}{Z_2}$.

Solution: The transformation $u = bz_1/z_2$ and $v = z_2$ has inverse $z_1 = uv/b$ and $z_2 = v$ giving Jacobian determinant $|J| = |v|/b$. The joint pdf of U, V therefore is $f_{U,V}(u,v) = \frac{|v|}{2\pi b} e^{-\frac{u^2v^2}{2b^2}} e^{-\frac{v^2}{2}} = \frac{|v|}{2\pi b} e^{-\frac{v^2(u^2+b^2)}{2b^2}}$. It follows that the pdf of U is

$$f_U(u) = \int_{-\infty}^{\infty} \frac{|v|}{2\pi b} e^{-\frac{v^2(u^2+b^2)}{2b^2}} \, dv = \frac{2}{2\pi b} \int_0^{\infty} v e^{-\frac{v^2(u^2+b^2)}{2b^2}} \, dv = \frac{b}{\pi(u^2+b^2)} \quad -\infty < u < \infty,$$

which is the Cauchy distribution.

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5. Compute the probability that a number chosen uniformly at random from the set of positive divisors of 10^{99} is an integer multiple of 10^{88}.

Solution: The prime factorization of 10^{99} is $2^{99} \cdot 5^{99}$, so all divisors of 10^{99} have the form $2^a \cdot 5^b$ where a and b are integers with $0 \le a, b \le 99$. Since there are 100 choices for each of a and b, 10^{99} has 100^2 positive integer divisors. Of these, the multiples of $10^{88} = 2^{88} \cdot 5^{88}$ must satisfy the inequalities $88 \le a, b \le 99$. Thus, there are 12 choices for each of a and b, so 12^2 of the 100^2 divisors of 10^{99} are multiples of 10^{88}. Consequently, the desired probability is

$$\frac{12^2}{100^2} = \frac{9}{625}.$$

Department of Applied Mathematics and Statistics
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Introductory Examination–Summer Session
Morning Exam–Linear Algebra

Tuesday, August 22, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
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6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let A be a nonsingular $n \times n$ matrix with integer entries. Show that $A^{-1}b$ is an integer vector (i.e., all entries are integers) for every integral vector $b \in \mathbb{Z}^n$ if and only if $\det(A) = 1$ or -1.

Solution: If $\det(A) = 1$ or -1, then the result follows from Cramer's rule, or the cofactor formula for the inverse: $A^{-1} = \frac{C^T}{\det(A)}$; Since C^T also has integer entries and $\det(A) = 1$ or -1, A^{-1} has integer entries and so $A^{-1}b$ is an integer vector.

Conversely, if $A^{-1}b$ is integral for every integral b, $A^{-1}e^i$ is integral, where e^i is the i-th standard unit vector. Thus, every column of A^{-1} has integer entries. Therefore, $\det(A^{-1})$ is an integer. However, $\det(A)\det(A^{-1}) = \det(I) = 1$ and both $\det(A)$ and $\det(A^{-1})$ are integers. Thus, they have to be 1 or -1.

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2. Determine, with proof, the set S of all real solutions (x, y, z) to the following system of three equations as a function of a:

$$\begin{aligned} 3za^2 - 3a + x + y + 1 &= 0 \\ 3x - a - y + z(a^2+4) - 5 &= 0 \\ za^2 - a - 4x + 9y + 9 &= 0. \end{aligned}$$

Solution: Let A denote the coefficient matrix

$$A := \begin{bmatrix} 1 & 1 & 3a^2 \\ 3 & -1 & a^2+4 \\ -4 & 9 & a^2 \end{bmatrix},$$

let u denote the vector $u := [x, y, z]^T$, and let $v := [3a-1, a+5, a-9]^T$ denote the right-hand-side vector. Then the system can be written equivalently as $Au = v$.

The determinant of A equals $52(a^2-1)$. We break the solution into cases:

(a) If $a \notin \{-1, 1\}$, then $\det A \neq 0$. In this case

$$A^{-1} = [52(a^2-1)]^{-1} \begin{bmatrix} -10a^2 - 36 & 26a^2 & 4a^2+4 \\ -7a^2 - 16 & 13a^2 & 8a^2 - 4 \\ 23 & -13 & -4 \end{bmatrix}$$

and S is a singleton with unique element

$$u = A^{-1}v = \left[\frac{2a}{a+1}, -\frac{1}{a+1}, \frac{1}{a+1}\right]^T.$$

(b) If $a \in \{-1, 1\}$, then

$$A := \begin{bmatrix} 1 & 1 & 3 \\ 3 & -1 & 5 \\ -4 & 9 & 1 \end{bmatrix}.$$

In this case the system reduces to the following system of three equations in the two variables $(s, t) := (x + 2z, y + z)$:

$$\begin{aligned} s + t &= 3a - 1 \\ 3s - t &= a + 5 \\ -4s + 9t &= a - 9. \end{aligned}$$

It is easily checked that if $a = -1$ there is no solution (s, t) to this system, and hence $S = \emptyset$; and that if $a = 1$ there is a unique solution $(s, t) = (2, 0)$ to this system, and hence S is the infinite set $S = \{[2y + 2, y, -y]^T : y \in \mathbf{R}\}$.

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3. Let S and T be two subspaces in R^n. Show that if there exists a $n \times n$ matrix A such that $T \subset \{y : y = Ax, x \in S\}$ then $\dim(S) \geq \dim(T)$.

Solution: It is trivially true if $\dim(T) = 0$. Let we assume $\dim(T) = k > 0$ and let $\{y_1, \cdots y_k\}$ be a basis of T. It follows from the assumption that there exist $\{x_1, \cdots, x_k\}$ such that $y_i = Ax_i$. Suffices it to show that $\{x_i\}$ are linearly independent. This must be true otherwise we get the contradiction that if $\{x_i\}$ are linearly dependent then so are vectors $\{y_i\}$.

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4. An $n \times n$ matrix H is called a *Hadamard matrix* provided the entries of H consist only of $+1$ or -1 and $H^T H = nI$, where, of course, I is the $n \times n$ identity. The following are all examples of Hadamard matrices:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Let H be a Hadamard matrix. Prove: H^T is a Hadamard matrix.

Solution: H^T still contains only $+1$ and -1. Since $H^T H = nI$ it follows $H^{-1} = \frac{1}{n} H^T$. Therefore, $nI = nHH^{-1} = nH(\frac{1}{n} H^T) = HH^T$.

5. Let A and B be $n \times n$ matrices and suppose that $\{v_1, v_2, \ldots, v_n\}$ are linearly independent vectors that are eigenvectors for *both* A and B. (The associated eigenvalues may be different.)

   Prove that A and B commute.

   Solution: Let S be the $n \times n$ matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$. Then $S^{-1}AS = \Lambda_A$ and $S^{-1}BS = \Lambda_B$ where Λ_A, Λ_B are $n \times n$ diagonal matrices.

   We can therefore write $A = S\Lambda_A S^{-1}$ and $B = S\Lambda_B S^{-1}$. Since diagonal matrices commute we have

$$\begin{aligned} AB &= \left(S\Lambda_A S^{-1}\right)\left(S\Lambda_B S^{-1}\right) = S\Lambda_A\Lambda_B S^{-1} \\ &= S\Lambda_B\Lambda_A S^{-1} = \left(S\Lambda_B S^{-1}\right)\left(S\Lambda_A S^{-1}\right) = BA \end{aligned}$$

   as required.