

Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION—SUMMER SESSION
MORNING EXAM—REAL ANALYSIS

Monday, August 21, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is $2/3$ of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Compute

$$\int_0^{\infty} [x]e^{-x} dx,$$

where $[x]$ is the greatest integer less than or equal to x . Simplify completely.

2. Prove that, if $f : (-a, a) \rightarrow \mathbb{R}$ is C^2 (with $a > 0$), one has

$$f(x) = f(0) + \frac{1}{2}(f'(x) + f'(0))x + o(x^2)$$

near $x = 0$.

3. Suppose (X, d) is a *compact* metric space. Let $f : X \rightarrow X$ be such that

$$d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y \in X.$$

Show that there exists a unique $x^* \in X$ such that $f(x^*) = x^*$.

Hint: Consider the function $\phi(x) := d(x, f(x))$ for $x \in X$.

4. Calculate the Fourier coefficients $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) dx$ of the function

$$f(x) = \begin{cases} +1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Use your result to establish the values of the following infinite series:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}, \quad \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Carefully justify all steps.

Hint: Use Parseval's theorem.

5. Let m be a natural number and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that for every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda^m f(x)$ [Such functions are called homogeneous of degree m]. Show that for all $y \in \mathbb{R}^n$, we have

$$f(y) = \frac{1}{m} \sum_{i=1}^n y_i \left. \frac{\partial f}{\partial x_i} \right|_{x=y}.$$

Department of Applied Mathematics and Statistics
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INTRODUCTORY EXAMINATION—SUMMER SESSION
AFTERNOON EXAM—PROBABILITY

Monday, August 21, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is $2/3$ of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. This examination will begin at 1:30 PM and end at 4:30 PM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Assume that the random variables X and Y are jointly Gaussian, with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 1$ and $E(XY) = \rho$. Let $a \in \mathbb{R}$ be fixed. Prove that the conditional distribution of Y given $X \geq a$ has a probability density function given by

$$f(y|X \geq a) = \frac{1 - \Phi(a; \rho y, 1 - \rho^2) e^{-y^2/2}}{1 - \Phi(a; 0, 1)} \frac{1}{\sqrt{2\pi}}$$

where $\Phi(\cdot; m, \sigma^2)$ is the cumulative distribution function of a Gaussian variable with mean m and variance σ^2 .

2. In general, we know uncorrelated random variables are not necessarily independent. However, suppose X_1 and X_2 are Bernoulli random variable taking the values 0 and 1. Show that if X_1 and X_2 are uncorrelated then they are independent as well.
3. Distribute n balls independently and uniformly at random among n boxes. Let N_n denote the number of empty boxes. Show that for any $\varepsilon > 0$, there is a μ_n such that

$$P\left(\left|\frac{N_n}{n} - \mu_n\right| > \varepsilon\right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Be sure to compute $\lim_{n \rightarrow \infty} \mu_n$.

4. Let Z_1 and Z_2 be independent standard normal random variables and let $b > 0$ be a fixed constant. Find the pdf of $U = \frac{bZ_1}{Z_2}$.
5. Compute the probability that a number chosen uniformly at random from the set of positive divisors of 10^{99} is an integer multiple of 10^{88} .

Department of Applied Mathematics and Statistics
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INTRODUCTORY EXAMINATION—SUMMER SESSION
MORNING EXAM—LINEAR ALGEBRA

Tuesday, August 22, 2017

Instructions: Read carefully!

1. This **closed-book** examination consists of 5 problems, each worth 5 points. The passing grade is $2/3$ of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.
2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.
3. The problems have not been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.
4. Start your answer to each problem on a **NEW** sheet of paper. Write only on **ONE SIDE** of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your **NAME** and the **PROBLEM NUMBER** on each sheet.
5. This examination will begin at 8:30 AM and end at 11:30 AM. You may leave before then, but in that case you may not return.
6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.
7. **No calculators of any sort are needed or permitted.**

1. Let A be a nonsingular $n \times n$ matrix with integer entries. Show that $A^{-1}b$ is an integer vector (i.e., all entries are integers) for every integral vector $b \in \mathbb{Z}^n$ if and only if $\det(A) = 1$ or -1 .
2. Determine, with proof, the set S of all real solutions (x, y, z) to the following system of three equations as a function of a :

$$\begin{aligned} 3za^2 - 3a + x + y + 1 &= 0 \\ 3x - a - y + z(a^2 + 4) - 5 &= 0 \\ za^2 - a - 4x + 9y + 9 &= 0. \end{aligned}$$

3. Let S and T be two subspaces in R^n . Show that if there exists a $n \times n$ matrix A such that $T \subset \{y : y = Ax, x \in S\}$ then $\dim(S) \geq \dim(T)$.
4. An $n \times n$ matrix H is called a *Hadamard matrix* provided the entries of H consist only of $+1$ or -1 and $H^T H = nI$, where, of course, I is the $n \times n$ identity. The following are all examples of Hadamard matrices:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Let H be a Hadamard matrix. Prove: H^T is a Hadamard matrix.

5. Let A and B be $n \times n$ matrices and suppose that $\{v_1, v_2, \dots, v_n\}$ are linearly independent vectors that are eigenvectors for *both* A and B . (The associated eigenvalues may be different.)

Prove that A and B commute.