Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session

Friday, August 26, 2016

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let a be a nonzero vector in $\mathbb{R}^n$ with all its components nonnegative and let A be the $n \times n$ matrix each of whose columns is equal to a. Show that A has exactly one positive eigenvalue.

Solution: Because A is of rank one, A has $\lambda = 0$ as an eigenvalue with multiplicity $n-1$. Now

$$Aa = [a, a, \ldots, a]a = [a, a, \ldots, a] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \left(\sum_{i=1}^{n} a_i \right) a,$$

so $\sum_{i=1}^{n} a_i$ is the other eigenvalue with associated eigenvector a. Moreover, since all the components of a are nonnegative with at least one component positive (since a is nonzero), we see the eigenvalue $\sum_{i=1}^{n} a_i$ is positive.

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2. Justify that there exist unique $a, b, c \in \mathbb{R}$ minimizing

$$\int_{-1}^{1} (x^3 - a - bx - cx^2)^2 dx$$

and compute the optimal a, b, and c.

Solution: In the finite-dimensional space $\mathbb{R}_3[X]$ of polynomials of degree at most three, F equals

$$F(a, b, c) = \|X^3 - (a + bX + cX^2)\|^2$$

for the norm on $\mathbb{R}_3[X]$ defined by

$$\|p\|^2 = \int_{-1}^{1} p(x)^2 dx.$$

F is thus a strictly convex function and has a unique minimizer. It is characterized by the equations $(\partial F)/(\partial a) = (\partial F)/(\partial b) = (\partial F)/(\partial c) = 0$ which gives

$$\frac{\partial F}{\partial a} = 0 \Leftrightarrow \int_{-1}^{1} -2(x^3 - a - bx - cx^2)dx = 0 \Leftrightarrow 4a + \frac{4}{3}c = 0,$$
$$\frac{\partial F}{\partial b} = 0 \Leftrightarrow \int_{-1}^{1} -2x(x^3 - a - bx - cx^2)dx = 0 \Leftrightarrow -\frac{4}{5} + \frac{4}{3}b = 0,$$
$$\frac{\partial F}{\partial c} = 0 \Leftrightarrow \int_{-1}^{1} -2x^2(x^3 - a - bx - cx^2)dx = 0 \Leftrightarrow \frac{4}{3}a + \frac{4}{5}c = 0.$$

Thus the minimizer is given by the linear system:

$$\begin{cases} 3a + c = 0 \\ 5b - 3 = 0 \\ 5a + 3c = 0 \end{cases}$$

giving $a = c = 0$ and $b = 3/5$.

Solution 2: The problem amounts in minimizing the distance of X^3 to the subspace $\mathbb{R}_2[X]$. Since the norm is an Euclidean norm, the solution is therefore the orthogonal projection of X^3 on $\mathbb{R}_2[X]$. Now, we observe that

$$\begin{aligned} \langle X^3, 1 \rangle &= \int_{-1}^{1} x^3 dx = 0, \\ \langle X^3, X^2 \rangle &= \int_{-1}^{1} x^5 dx = 0, \end{aligned}$$

so X^3 is orthogonal to the subspace of $\mathbb{R}_2[X]$ generated by $(1, X^2)$. The projection of X^3 on $\mathbb{R}_2[X]$ is therefore equal to bX with

$$b = \frac{1}{\|X\|^2} \langle X^3, X \rangle = \left(\int_{-1}^{1} x^2 dx \right)^{-1} \left(\int_{-1}^{1} x^4 dx \right) = \frac{3}{2} \times \frac{2}{5} = \frac{3}{5}.$$

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3. Cards are drawn one by one, at random, and without replacement, from a standard deck of 52 playing cards. What is the probability that the fourth Heart is drawn on the tenth draw? (Do not simplify to a decimal number.)

Solution: Let A be the event that there are exactly three Hearts drawn in the first nine draws, and let B be the event that the tenth draw is a Heart.

Rephrasing its description, A is the event that in choosing 9 cards, 3 are chosen from the 13 Hearts and 6 from the 39 non-Hearts, so using an equally-likely probability model, we have

$$P[A] = \frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}}.$$

Conditional upon event A, B is the event that a randomly chosen card drawn from a (partial) deck of 10 Hearts and 33 non-Hearts is a Heart. Thus,

$$P[B|A] = \frac{10}{43}.$$

By the multiplication rule for the probability of an intersection event,

$$P[AB] = P[B|A]P[A] = \frac{10}{43}\frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}} = \frac{27417}{464830}.$$

Alternatively, randomly pick 10 cards, one at a time, from the deck. Let E be the event that 4 of the 10 are hearts, and let F be the event that the last card is a heart. Given E, all $\binom{10}{4}$ positions for the four hearts are equally likely, and $\binom{9}{3}$ of the possible positions have a heart appearing last. Consequently,

$$P[F|E] = \frac{\binom{9}{3}}{\binom{10}{4}} = \frac{4}{10}.$$

Thus

$$P[E \cap F] = P[E] \times P[F|E] = \frac{\binom{13}{4}\binom{39}{6}}{\binom{52}{10}} \times \frac{4}{10} = \frac{27417}{464830}.$$

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4. Show that for any three real numbers a, b, and c, the following inequality holds:

$$\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{6}\right)^2 \le \frac{a^2}{2} + \frac{b^2}{3} + \frac{c^2}{6}.$$

Solution: We apply Cauchy–Schwartz inequality to get

$$\begin{aligned}
\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{6}\right)^2 &= \left(\frac{1}{\sqrt{2}} \cdot \frac{a}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} + \frac{1}{\sqrt{6}} \cdot \frac{c}{\sqrt{6}}\right)^2 \\
&\le \left(\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2\right)\left(\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{b}{\sqrt{3}}\right)^2 + \left(\frac{c}{\sqrt{6}}\right)^2\right) \\
&= \frac{a^2}{2} + \frac{b^2}{3} + \frac{c^2}{6}.
\end{aligned}$$

Alternatively, consider a random variable X taking values a, b, and c with probabilities $1/2$, $1/3$, and $1/6$, respectively. X has a nonnegative variance so $E[X]^2 \le E[X^2]$.

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5. (a) Show that if a complex matrix $A \in \mathbf{C}^{n\times n}$ satisfies $x^*Ax = 0$ for all complex vectors $x \in \mathbf{C}^n$, then A is a zero matrix.

   (b) Show that if $n \geq 2$, then there exists a nonzero real matrix $A \in \mathbf{R}^{n\times n}$ such that $x^TAx = 0$ for all real vectors $x \in \mathbf{R}^n$.

Solution: (a) Let $A \in \mathbf{C}^{n\times n}$ be a matrix satisfying $x^*Ax = 0$ for all $x \in \mathbf{C}^n$. Let $x, y \in \mathbf{C}^n$ be two arbitrary vectors. Then both $(x+y)^*A(x+y)$ and $(x+iy)^*A(x+iy)$ are zero by the assumption. Combining this with $x^*Ax = y^*Ay = 0$, we get $x^*Ay = 0$. Since x and y can be chosen arbitrarily, one can choose $x = e_i$, $y = e_j$, for any $i, j = 1, \ldots, n$, where e_k is the k-th standard unit vector in $\mathbf{C}^n$. Since $e_i^*Ae_j = a_{ij}$, the (i, j)-th entry of A, we get $a_{ij} = 0$, for all $i, j = 1, \ldots, n$. In other words, $A = 0$.

(b) Since $n \geq 2$, there exists a nonzero real $A \in \mathbf{R}^{n\times n}$ s.t. $A = -A^T$. One possible choice for A is $A = [a_{ij}]$, with $a_{12} = -1$, $a_{21} = 1$, and $a_{ij} = 0$ elsewhere. For any matrix $A \in \mathbf{R}^{n\times n}$ with $A = -A^T$, we have $x^TAx = 0$ since the scalar x^TAx satisfies

$$x^TAx = (x^TAx)^T = x^TA^Tx = x^T(-A)x = -x^TAx.$$

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6. Let X_1 and X_2 be two iid random variables with the uniform distribution over $[0, 1]$. Define $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Compute the covariance $\text{cov}(Y, Z)$.

Solution: First, we observe that

$$P\{Z \leq z\} = P\{X_1 \leq z\}P\{X_2 \leq z\} = z^2,$$

which implies Z has probability density $2z$, for $z \in [0, 1]$. This in turns yields the expectation $E\{Z\} = \frac{2}{3}$. As for the minimum Y, we deduce that

$$P\{Y \geq y\} = P\{X_1 \geq y\}^2 = (1 - y)^2,$$

which implies through integration that $\mathbb{E}\{Y\} = \frac{1}{3}$.

Since $YZ = X_1X_2$ we have

$$\mathbb{E}\{YZ\} = \mathbb{E}\{X_1X_2\} = \frac{1}{4}.$$

Finally, the covariance is given by

$$\text{cov}(Y, Z) = \mathbb{E}\{YZ\} - \mathbb{E}\{Y\}\mathbb{E}\{Z\} = \frac{1}{36}.$$

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7. Let $f : [a, b] \to \mathbb{R}$ be such that the sets $\{x : f(x) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$ (such a function is called upper semi-continuous). Prove that f has a maximizer, i.e., that there exists $x_0 \in [a, b]$ such that $f(x) \leq f(x_0)$ for all $x \in [a, b]$.

Solution: We may assume $a < b$. Let $\gamma = \sup_{x\in[a,b]} f(x) \leq +\infty$, and let y_n be a strictly increasing sequence that converges to γ. The set $F_n = \{x \in [a, b] \ : \ f(x) \geq y_n\}$ is, by assumption, a closed set, and is nonempty by the denition of the supremum. Being closed subsets of the compact set $[a, b]$, the F_n are also compact. Thus, we have a nested sequence $F_1 \supseteq F_2 \supseteq F_3 \cdots$ of nonempty compact sets, so by the Cantor intersection theorem the intersection $\bigcap_{i=1}^{n} F_i$ is also nonempty. If x is any element of this intersection, then $f(x) \geq y_n$ for all n, which is only possible if γ is finite. It follows that $f(x) \geq \lim_{n\to\infty} y_n = \gamma \geq f(x)$, hence $f(x) = \gamma$. This proves that x is a maximizer of f.

Solution 2: Let $\gamma = \sup_{x\in[a,b]} f(x)$ and take $(x_n)_{n\in\mathbb{N}}$ to be a sequence in $[a, b]$ such that $f(x_n) \to \gamma$. Since $[a, b]$ is compact, we can, if necessary, replace the sequence by a convergent subsequence, so we can assume $x_n \to x^*$ for some $x^* \in [a, b]$. By upper-semi-continuity, for any $\epsilon > 0$ the set

$$\{x \in [a, b] \ : \ f(x) < f(x^*) + \epsilon\}$$

is an open set containing x^*. Thus, there exists $\delta > 0$ such that $f(x) < f(x^*) + \epsilon$ for all $|x - x^*| < \delta$. Consequently,

$$\gamma = \limsup_{n\to\infty} f(x_n) \leq f(x^*).$$

We conclude that $\gamma < +\infty$ and $f(x^*) = \gamma$, which shows that x^* is a maximizer of f.

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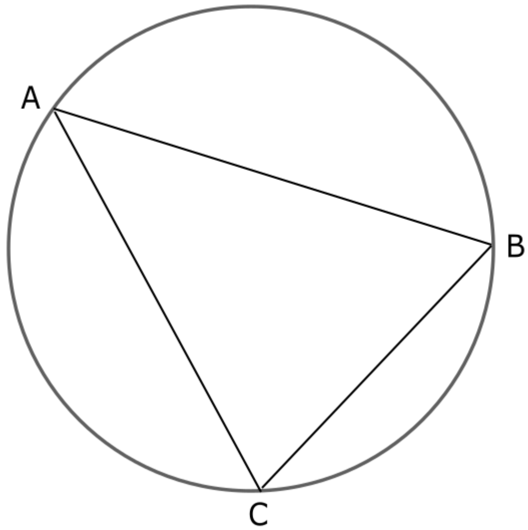
8. Three points, A, B, and C, are chosen uniformly and independently on a circle of radius 1.



What is the expected perimeter of triangle ABC?

Hint: Try first to compute the expected value of the length of AB.

Solution: Without loss of generality, place A at $(1, 0)$ and B at $(\cos t, \sin t)$ for $t \in [0, 2\pi]$. The length of segment AB is

$$\sqrt{(\cos t - 1)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

Therefore the expected length of this segment is

$$\frac{1}{2\pi} \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \frac{8}{2\pi} = \frac{4}{\pi}.$$

Finally, by linearity of expectation, the expected perimeter of the triangle is $12/\pi$.

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9. Let n be a positive integer and let A_n be the $n \times n$ matrix with entries as follows:

- All entries above the diagonal are 1.
- All entries on the diagonal are 0.
- All entries below the diagonal are -1.

For example,

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Prove that $\det(A_n) = 0$ for n odd and $\det(A_n) = 1$ for n even.

Hint: Start by adding the last column to the first.

Solution: For $n = 1$ or 2 we have

$$\det A_1 = \det[0] = 0 \qquad \text{and} \qquad \det A_2 = \det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 1.$$

For $n > 2$ we can write A_n in block form:

$$A_n = \begin{bmatrix} 0 & \mathbf{1}_{n-2}^T & 1 \\ -\mathbf{1}_{n-2} & A_{n-2} & \mathbf{1}_{n-2} \\ -1 & -\mathbf{1}_{n-2}^T & 0 \end{bmatrix},$$

where $\mathbf{1}_{n-2}$ denotes the $(n-2) \times 1$ vector all of whose entries are 1.

Using the hint, we add the last column of A_n to the first (this doesn't change the determinant):

$$\det A_n = \det \begin{bmatrix} 1 & \mathbf{1}_{n-2}^T & 1 \\ \mathbf{0}_{n-2} & A_{n-2} & \mathbf{1}_{n-2} \\ -1 & -\mathbf{1}_{n-2}^T & 0 \end{bmatrix},$$

where $\mathbf{0}_{n-2}$ denotes the $(n-2) \times 1$ vector all of whose entries are 0. Next add the first row of the result to the last:

$$\det A_n = \det \begin{bmatrix} 1 & \mathbf{1}_{n-1}^T & 1 \\ \mathbf{0}_{n-1} & A_{n-2} & \mathbf{1}_{n-1} \\ 0 & \mathbf{0}_{n-1}^T & 1 \end{bmatrix}.$$

Cofactor expansion of this latter matrix down the first column (and then across the last row) gives $\det A_n = \det A_{n-2}$ and the result follows by induction.

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10. Let $\Omega = (\omega_{ij})$ be a symmetric real $n \times n$ matrix with the property that $\Omega \mathbf{1}_n = 0$, where $\mathbf{1}_n$ denotes the $n \times 1$ vector all of whose entries are 1, and write

$$\Omega = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix},$$

where A is $(n-1)\times(n-1)$, b is $(n-1)\times 1$, and c is a scalar. If we define

$$\Gamma = A - b\mathbf{1}_{n-1}^T - \mathbf{1}_{n-1}b^T + c\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T.$$

show that

$$\Gamma = (I+J)A(I+J),$$

where J denotes the $(n-1)\times(n-1)$ matrix all of whose entries are 1. If A is positive definite can we conclude that Γ is positive definite as well? Justify your answer.

Hint: First use the fact that $\Omega\mathbf{1}_n = 0$ to find expressions for b and c in terms of A.

Solution: Since $\Omega\mathbf{1}_n = 0$, we have

$$\begin{pmatrix} A & b \\ b^T & c \end{pmatrix}\begin{pmatrix} \mathbf{1}_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} A\mathbf{1}_{n-1} + b \\ b^T\mathbf{1}_{n-1} + c \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n-1} \\ 0 \end{pmatrix},$$

and this leads to $b = -A\mathbf{1}_{n-1}$ and $c = \mathbf{1}_{n-1}^T A\mathbf{1}_{n-1}$. Observe that c is the sum of the entries of A, and that $c\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T = JAJ$.

Substituting the expressions for b and c into the definition of Γ gives

$$\Gamma = A + A\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T + \mathbf{1}_{n-1}\mathbf{1}_{n-1}^T A + (\mathbf{1}_{n-1}^T A\mathbf{1}_{n-1})\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T.$$

$$= A + AJ + JA + JAJ = (I+J)A(I+J).$$

If A is positive definite, so is Γ. To see this, for any non-zero $(n-1)$-vector x we have $x^TAx > 0$. In addition, since the matrix $I+J$ is invertible (it's inverse is $I-(1/n)J$), letting $y = (I+J)x$, we have that y is non-zero, so

$$x^T\Gamma x = x^T(I+J)A(I+J)x = y^TAy > 0.$$

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11. Let f be a uniformly continuous function on a finite interval (a,b). Is it true that f must be bounded on (a,b), that is $\sup_{(a,b)}|f(x)| < \infty$? If true, prove it; otherwise, give a counterexample.

Solution: The statement is true. If we take any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in (a,b)$ with $|x-y| \le \delta$ we have $|f(x) - f(y)| \le \epsilon$. Taking a finite sequence of points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with $x_{i+1} - x_i \le \delta$, we have

$$|f(x) - f(y)| \le \epsilon$$

for all $x, y \in [x_i, x_{i+1}]$ and $i = 0, \dots, n-1$ when $x \neq a$ and $y \neq b$. Note that as a consequence $|f(x_{i+1}) - f(x_i)| \leq \epsilon$ for $i = 1, 2, 3, \dots, n-2$.

To complete the proof, we show $|f(x)| \leq |f(x_1)| + n\epsilon$ for all $x \in (a, b)$. First, if $x \in (a, x_1)$, then

$$\begin{aligned} |f(x)| &= |f(x_1) - f(x) + f(x_1)| \\ &\leq |f(x_1)| + |f(x_1) - f(x)| \leq |f(x_1)| + \epsilon \\ &\leq |f(x_1)| + n\epsilon \end{aligned}$$

Next, if $x \in [x_i, x_{i+1})$ with $1 \leq i \leq n-1$ we have

$$\begin{aligned} |f(x)| &= |f(x_1) + [f(x_2) - f(x_1)] + [f(x_3) - f(x_2)] + \dots + [f(x) - f(x_i)]| \\ &\leq |f(x_1)| + i\epsilon \leq |f(x_1)| + n\epsilon. \end{aligned}$$

Solution 2: Let $\varepsilon > 0$. Since $f : (a, b) \to \mathbb{R}$ is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in (a, b)$ with $|x - y| < \delta$. Without loss of generality, we assume $\delta < (b - a)/2$. Now f restricted to the closed interval $[a + \frac{\delta}{2}, b - \frac{\delta}{2}]$ is bounded, say, by M. Moreover, if $x \in (a, a + \frac{\delta}{2}]$, then $|f(x)| \leq |f(x) - f(a + \frac{\delta}{2})| + |f(a + \frac{\delta}{2})| < \varepsilon + M$; similarly, if $x \in [b - \frac{\delta}{2}, b)$, $|f(x)| \leq |f(x) - f(b - \frac{\delta}{2})| + |f(b - \frac{\delta}{2})| < \varepsilon + M$. This shows f is bounded on the interval (a, b).

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12. Let A be the $n \times n$ matrix

$$\begin{pmatrix} 1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Find an invertible $n \times n$ matrix B such that $BAB^{-1} = A^T$.

Hint: Take B to be a permutation matrix.

Solution: Observe that a_{ij} depends only on the difference $i - j$, so

$$(A^T)_{i,j} = a_{j,i} = a_{n+1-i,n+1-j}.$$

Take $B = (b_{i,j})$ to be the permutation matrix corresponding to the permutation that reverses order, i.e.,

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

i.e.,

$$b_{i,j} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise .} \end{cases}$$

Then

$$(BB)_{i,j} = \sum_k b_{i,k} b_{k,j} = \sum_{k:k=n+1-i,n+1-j} b_{i,k} b_{k,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

that is, $B = B^{-1}$. Furthermore, we have

$$(AB)_{i,j} = \sum_k a_{i,k} b_{k,j} = \sum_{k=n+1-j} a_{ik} = a_{i,n+1-j},$$

the matrix obtained by reversing the columns of A. Similarly, BA is the matrix obtained by reversing the rows of A. Thus,

$$(BAB^{-1})_{ij} = a_{n+1-i,n+1-j} = a_{j,i}.$$

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13. Consider n people $S_1, \dots, S_n$. S_1 receives a binary information "yes" or "no" and transmits it to S_2, who transmits it to S_3, and so on, until person S_n. Each person transmits the information that he/she hears with probability p and the opposite information with probability $1 - p$, independently from the others. Denote by A_i the event "person i transmits the initial information" and by p_i its probability. Find a recursion relation between p_i and p_{i-1} and deduce the probability p_n that the right information is received by the last person. What happens to p_n as $n \to \infty$ for $p \in (0, 1)$?

    Hint: Replace the recursion for p_n by a recursion for $p_n - \frac{1}{2}$.

Solution: We can write

$$P[A_i | A_{i-1}] = p,$$

and

$$P[A_i|A_{i-1}^c] = 1 - p,$$

so

$$\begin{aligned} p_i &= P[A_i] = pP[A_{i-1}] + (1-p)P[A_{i-1}^c] \\ &= pp_{i-1} + (1-p)(1-p_{i-1}) \\ &= (2p-1)p_{i-1} + 1 - p. \end{aligned}$$

Subtracting $\frac{1}{2}$ from both sides we obtain

$$p_i - \frac{1}{2} = (2p-1)p_{i-1} + 1 - p - \frac{1}{2} = (2p-1)(p_{i-1} - \frac{1}{2}).$$

Consequently

$$p_n - \frac{1}{2} = (2p-1)^{n-1}(p_1 - \frac{1}{2}) = (2p-1)^{n-1}(p - \frac{1}{2}),\ n = 1, 2, 3, \ldots.$$

For any choice of $p \in (0,1)$ we see that $p_n - \frac{1}{2} \to 0$ as $n \to \infty$ so $p_n \to \frac{1}{2}$.

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14. Laurel and Hardy are planning to meet between 5pm and 6pm. They agree that each one, when he gets there, will wait for the other for at most 10 minutes. Assuming that they arrive at the meeting point independently and at times uniformly distributed between 5pm and 6pm, what is the probability that they manage to meet?

Solution: Let's denote by X and Y the random variables corresponding to Laurel and Hardy's arrival times. We can simplify the problem a little bit by assuming that X and Y are defined on the interval $[0, 1]$, with 0 corresponding to 5pm and 1 to 6pm. Then, the probability of meeting is the probability of the event $A = \{|X - Y| \le \frac{1}{6}\}$. Let's consider the complementary event A^c:

$$\begin{aligned} P(A^c) = P(|X-Y| > \frac{1}{6}) &= P(Y > X + \frac{1}{6}) + P(Y < X - \frac{1}{6}) \\ &= \int_0^{1-\frac{1}{6}} \int_{x+\frac{1}{6}}^1 dy\, dx + \int_{\frac{1}{6}}^1 \int_0^{x-\frac{1}{6}} dy\, dx \end{aligned}$$

These two integrals correspond to the areas of two right triangles of sides $1 - \frac{1}{6}$ and $1 - \frac{1}{6}$; therefore $P(A^c) = (1 - \frac{1}{6})^2$, which gives $P(A) = 11/36 \approx 0.31$.

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15. Let f be a real-valued continuous function defined on the interval $[a, b]$. Suppose that

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt \quad \text{for all} \quad x \in [a, b].$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution: Define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\,dt - \int_x^b f(t)\,dt.$$

It then follows that $F(x) = 0$ for all $x \in [a, b]$. Combining this fact with the fundamental theorem of calculus allows us to conclude for all $x \in (a, b)$ that

$$0 = F'(x) = 2f(x)$$

Finally, by continuity $f(x) = 0$ for $x = a, b$.