Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session

Friday, August 26, 2016

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Let a be a nonzero vector in $\mathbb{R}^n$ with all its components nonnegative and let A be the $n \times n$ matrix each of whose columns is equal to a. Show that A has exactly one positive eigenvalue.

2. Justify that there exist unique $a, b, c \in \mathbb{R}$ minimizing

$$\int_{-1}^{1} (x^3 - a - bx - cx^2)^2 dx$$

and compute the optimal a, b, and c.

3. Cards are drawn one by one, at random, and without replacement, from a standard deck of 52 playing cards. What is the probability that the fourth Heart is drawn on the tenth draw? (Do not simplify to a decimal number.)

4. Show that for any three real numbers a, b, and c, the following inequality holds:

$$\left(\frac{a}{2} + \frac{b}{3} + \frac{c}{6}\right)^2 \leq \frac{a^2}{2} + \frac{b^2}{3} + \frac{c^2}{6}.$$

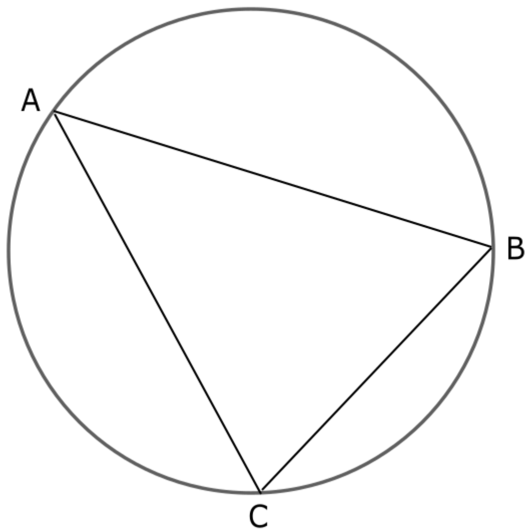
5. (a) Show that if a complex matrix $A \in \mathbf{C}^{n \times n}$ satisfies $x^* A x = 0$ for all complex vectors $x \in \mathbf{C}^n$, then A is a zero matrix.

   (b) Show that if $n \geq 2$, then there exists a nonzero real matrix $A \in \mathbf{R}^{n \times n}$ such that $x^T A x = 0$ for all real vectors $x \in \mathbf{R}^n$.

6. Let X_1 and X_2 be two iid random variables with the uniform distribution over $[0, 1]$. Define $Y = \min(X_1, X_2)$ and $Z = \max(X_1, X_2)$. Compute the covariance $\text{cov}(Y, Z)$.

7. Let $f : [a, b] \to \mathbb{R}$ be such that the sets $\{x : f(x) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$ (such a function is called upper semi-continuous). Prove that f has a maximizer, i.e., that there exists $x_0 \in [a, b]$ such that $f(x) \leq f(x_0)$ for all $x \in [a, b]$.

8. Three points, A, B, and C, are chosen uniformly and independently on a circle of radius 1.



What is the expected perimeter of triangle ABC?

Hint: Try first to compute the expected value of the length of AB.

9. Let n be a positive integer and let A_n be the $n \times n$ matrix with entries as follows:

- All entries above the diagonal are 1.
- All entries on the diagonal are 0.
- All entries below the diagonal are -1.

For example,

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Prove that $\det(A_n) = 0$ for n odd and $\det(A_n) = 1$ for n even.

Hint: Start by adding the last column to the first.

10. Let $\Omega = (\omega_{ij})$ be a symmetric real $n \times n$ matrix with the property that $\Omega \mathbf{1}_n = 0$, where $\mathbf{1}_n$ denotes the $n \times 1$ vector all of whose entries are 1, and write

$$\Omega = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix},$$

where A is $(n-1) \times (n-1)$, b is $(n-1) \times 1$, and c is a scalar. If we define

$$\Gamma = A - b\mathbf{1}_{n-1}^T - \mathbf{1}_{n-1}b^T + c\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T.$$

show that

$$\Gamma = (I + J)A(I + J),$$

where J denotes the $(n-1) \times (n-1)$ matrix all of whose entries are 1. If A is positive definite can we conclude that Γ is positive definite as well? Justify your answer.

Hint: First use the fact that $\Omega \mathbf{1}_n = 0$ to find expressions for b and c in terms of A.

11. Let f be a uniformly continuous function on a finite interval (a, b). Is it true that f must be bounded on (a, b), that is $\sup_{(a,b)} |f(x)| < \infty$? If true, prove it; otherwise, give a counterexample.

12. Let A be the $n \times n$ matrix

$$\begin{pmatrix} 1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Find an invertible $n \times n$ matrix B such that $BAB^{-1} = A^T$.

Hint: Take B to be a permutation matrix.

13. Consider n people $S_1, \dots, S_n$. S_1 receives a binary information "yes" or "no" and transmits it to S_2, who transmits it to S_3, and so on, until person S_n. Each person transmits the information that he/she hears with probability p and the opposite information with probability $1-p$, independently from the others. Denote by A_i the event "person i transmits the initial information" and by p_i its probability. Find a

recursion relation between p_i and p_{i-1} and deduce the probability p_n that the right information is received by the last person. What happens to p_n as $n \to \infty$ for $p \in (0, 1)$?

Hint: Replace the recursion for p_n by a recursion for $p_n - \frac{1}{2}$.

14. Laurel and Hardy are planning to meet between 5pm and 6pm. They agree that each one, when he gets there, will wait for the other for at most 10 minutes. Assuming that they arrive at the meeting point independently and at times uniformly distributed between 5pm and 6pm, what is the probability that they manage to meet?

15. Let f be a real-valued continuous function defined on the interval $[a, b]$. Suppose that

$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt \quad \text{for all} \quad x \in [a, b].$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.