Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Summer Session

Friday, August 22, 2014

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

Solution: Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \frac{\pi^2}{24} = \frac{\pi^2}{6}$, it follows that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

---

2. Suppose F is a closed proper subset of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ such that $x_0 \notin F$. Prove that there must exist two *disjoint* open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $x_0 \in \mathcal{O}_1$ and $F \subset \mathcal{O}_2$.

Solution: Since $x_0 \in F^c$ and F^c is open, there is a $d > 0$ such that $B_d(x_0) := \{x : |x - x_0| < d\}$ is contained in F^c. This shows $|y - x_0| \geq d$ for every $y \in F$. Now, define $\mathcal{O}_1 = \{x : |x - x_0| < d/2\}$, and $\mathcal{O}_2 = \{x : |x - x_0| > d/2\}$. $\mathcal{O}_1$ and $\mathcal{O}_2$ are clearly open, and they are disjoint since a point $x \in \mathbb{R}^n$ cannot simultaneously be less than and greater than $d/2$ from x_0. Notice that $x_0 \in \mathcal{O}_1$ since $|x_0 - x_0| = 0 < d/2$. Moreover, if $y \in F$ then $|y - x_0| \geq d > d/2$, which shows $F \subset \mathcal{O}_2$.

---

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an n-th degree polynomial which has n distinct real roots, where $n \geq 2$. Show that its derivative f' has $n - 1$ distinct real roots.

Solution: Denote the roots of f by $r_1 < r_2 < r_3 < \cdots < r_n$.

For each $i = 2, 3, 4, \ldots, n$, either $f(x) > 0$ for all $x \in (r_{[i-1]}, r_i)$ or $f(x) < 0$ for all $x \in (r_{[i-1]}, r_i)$. (Otherwise, by continuity of f there would be a root of f in the interior of each such interval.)

Since f is continuous, it attains a maximum or minimum, respectively, at least once in the interior of the interval.

Since f is differentiable everywhere, at each such maximum or minimum point x, $f'(x) = 0$. Thus, there is at least one real root of f' in each of the $n-1$ intervals.

However, f' is an $(n-1)$-degree polynomial, so has exactly $n-1$ roots. Thus, there is only one root of f' in each of the $n-1$ intervals. So, all roots of the derivative are real and distinct.

---

4. (a) Prove that, for all $t \geq 0$, the value of the integral

$$f(t) = \int_0^\infty \frac{1 - \exp(-tx^2)}{x^2}\, dx$$

is a finite real number and that $f(t) = f(1)\sqrt{t}$ with $f(1) > 0$.

(b) Justifying each step of your calculation, compute $f'(t)$ explicitly and deduce from it the expression of $f(t)$ for all t.

Solution:

(a) We need only show that the integrand is suitably well behaved near the origin and near ∞. Indeed, the integrand approaches t as $x \downarrow 0$ and is asymptotically equivalent to x^{-2} (which is integrable at ∞) as $x \to \infty$. If $t = 0$, the integral is zero, and for any $t > 0$, the integrand is positive and so is the integral. For $t > 0$, making the change of variable $y = \sqrt{t}x$ yields

$$f(t) = \sqrt{t} \int_0^\infty \frac{1 - \exp(-y^2)}{y^2}\, dy = f(1)\sqrt{t}$$

and this is also valid for $t = 0$.

(b) One can justify differentiation (with respect to t) under the integral by standard theorems in the theory of improper Riemann integrals; a somewhat more sophisticated approach applies the dominated convergence theorem to difference quotients. In any case, we find, for $t > 0$,

$$f'(t) = \int_0^\infty e^{-tx^2}\, dx = \tfrac{1}{2}\sqrt{\pi}\, t^{-1/2}.$$

Therefore

$$f(t) = \sqrt{\pi t} + c$$

for some constant c. Comparing to the previous result, we have $c = 0$ and $f(t) = \sqrt{\pi t}$.

---

5. Suppose that $f : [0, \infty) \to \mathbb{R}$ satisfies

$$f(1) = 1 \quad \text{and} \quad f'(x) = \frac{1}{x^2 + f(x)^2} \quad \text{for } x \geq 1.$$

Prove that $\lim_{x\to\infty} f(x)$ exists and satisfies $\lim_{x\to\infty} f(x) \leq 1 + \pi/4$.

Solution: It is easy to see that $f'(x) > 0$ for all $x \in [1, \infty)$. This may be combined with $f(1) = 1$ to conclude that f is strictly monotonically increasing on $[1, \infty)$ and, in particular, that

$$f(x) \geq f(1) = 1 \text{ for all } x \geq 1. \tag{1}$$

From the Fundamental Theorem of Calculus, (5), and (1) we have

$$\begin{aligned} f(x) = f(1) + \int_1^x f'(t)dt = 1 + \int_1^x f'(t)dt = 1 + \int_1^x \frac{1}{t^2 + f(t)^2} \\ \leq 1 + \int_1^x \frac{1}{t^2 + 1} \\ = 1 + \left[\tan^{-1}(t)\right]_1^x = 1 + \tan^{-1}(x) - \tan^{-1}(1) = 1 - \frac{\pi}{4} + \tan^{-1}(x). \end{aligned}$$

It then easily follows that

$$\begin{aligned} \limsup_{x\geq 1} f(x) \leq \limsup_{x\geq 1} \left(1 - \frac{\pi}{4} + \tan^{-1}(x)\right) \\ = \lim_{x\to\infty} \left(1 - \frac{\pi}{4} + \tan^{-1}(x)\right) = 1 - \frac{\pi}{4} + \frac{\pi}{2} = 1 + \frac{\pi}{4}. \end{aligned}$$

Thus, since $f(x)$ is monotonically increasing, we know that $\lim_{k\to\infty} f(x)$ exists and is bounded by $1 + \frac{\pi}{4}$.

---

6. Find a 2×2 matrix A with real entries such that $A^3 = I$ and $A \neq I$.

Solution: Rotation by $2\pi/3$ works:

$$\begin{bmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$$

---

7. Let $x, y \in \mathbb{R}^n$, and let A be an $n \times n$ real matrix. Assume that A is orthogonal, and show that $|x - y| = |Ax - Ay|$.

Solution: Since A orthogonal implies $A^T A = I$, we have $|x-y|^2 = (x-y)^T(x-y) = (x-y)^T(A^TA)(x-y) = (A(x-y))^T(A(x-y)) = |Ax - Ay|^2$.

---

8. Let F, G and H be three vector subspaces of a vector space E. Assume that the union is also a vector subspace. Prove that one of F, G or H contains the other two. Provide an "algebraic" proof of the result, avoiding any topological argument (no limit, density, closed or open sets etc.).

   Hint: It may be helpful to prove the result for two subspaces first.

Solution: The case of two subspaces is straightforward. If $F \cup G$ is a vector space, and there exists f in F but not in G, then for any $g \in G$, one has $f + g \in F \cup G$, but not in G (because $f \in G$ otherwise), which implies that $f + g \in F$ and therefore $g \in F$. Consequently $G \subset F$.

Consider three subspaces. If any of them is included in the union of the other two, we are done by the previous argument. Assume then that $H \not\subset G \cup F$. Take $h \in H, h \notin G \cup F$. If $g \in G$, then $h + g, h - g \in F \cup G \cup H$. If either $h + g$ or $h - g$ belongs to G, then h belongs to G too, which is a contradiction. If the two belong to F, then so does their sum, which implies that $h \in F$, another contradiction. So, one of them must belong to H, which implies that $g \in H$. Since this is true for all $g \in G$, we get $G \subset H$, and the same argument proves that $F \subset H$.

---

9. Define $\|A\|_2 = \sqrt{\sum_{k,l=1}^n a_{kl}^2}$. Let A be a $n \times n$ real matrix with $\|A\|_2 < 1$, show that $(I + A)^{-1}$ exists and

$$\|(I + A)^{-1}\|_2 \leq \frac{\sqrt{n}}{1 - \|A\|_2}.$$

Solution: One has

$$|Ax| = \sqrt{\sum_{k=1}^n \Big(\sum_{l=1}^n a_{kl} x_l\Big)^2} \leq \sqrt{\sum_{k=1}^n \Big(\sum_{l=1}^n a_{kl}^2 |x|^2\Big)}$$

by Schwartz inequality. This yields $|Ax| \leq \|A\|_2 |x|$. More generally, the same argument proves that $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.

This implies, in particular, that every eigenvalue λ of A satisfies $|\lambda| < 1$. Therefore, for any eigenvalue of $I + A$, which is of the form $1 + \lambda$, we have $|1 + \lambda| \geq 1 - |\lambda| > 0$. Therefore, no eigenvalue of $I + A$ can be zero and hence $I + A$ is nonsingular.

For the second part, from $(I + A)(I + A)^{-1} = I$, we have that

$$(I + A)^{-1} = I - A(I + A)^{-1}.$$

Taking norm of both sides, we get

$$\|(I + A)^{-1}\|_2 \leq \|I\|_2 + \|A\|_2 \|(I + A)^{-1}\|_2$$

and hence,

$$(1 - \|A\|_2)\|(I + A)^{-1}\|_2 \leq \|I\|_2 = \sqrt{n}.$$

---

10. Let A be a symmetric $n \times n$ matrix. Show that there exists a natural number $k \geq 1$ such that $A^k = 0$ if and only if $A = 0$.

Solution: Since A is symmetric, we can decompose $A = SDS^{-1}$ where S is the matrix of eigenvectors and D is a diagonal matrix of the eigenvalues. $A^k = 0$ if and only if $SD^kS^{-1} = 0$ if and only if $D^k = 0$ if and only if $D = 0$ if and only if $A = 0$.

---

11. If n men, among whom are A and B, stand in a row arranged at random, what is the probability that there will be exactly r men between A and B, $r = 0, 1, ..., n - 2$? Same question if they stand in a ring instead of in a row. (In the circular arrangement consider only the arc leading from A to B in the anticlockwise direction.)

Solution: Let R be the number of men between A and B. For the linear case, there are $n(n - 1)$ ways to place A and B. Of these, assuming A is to the left of B, there are $n - r - 1$ ways to place A so that $R = r$, and the same for B to the left of A. Hence, $P(R = r) = \frac{2(n-r-1)}{n(n-1)}$. In the case of a circle, all spacings are equally likely, so $P(R = r) = \frac{1}{n-1}, r = 0, 1, ..., n - 2$.

---

12. Let X denote the number of different days of a 365-day year that are birthdays of four persons selected randomly. Calculate $E[X]$.

Solution: With the simplifying assumption that there are 365 days in a year, denoted 1 through 365, let I_k denote the indicator of the event that day k, $k = 1, 2, 3, \ldots, 365$, is the birthday of at least one of the persons. Then

$$X = I_1 + I_2 + \cdots + I_{365}$$

and by linearity of expectation

$$E[X] = E[I_1 + I_2 + \cdots + I_{365}] = E[I_1] + E[I_2] + \cdots + E[I_{365}].$$

With the modeling assumptions that the persons' birthdays are independent and equally likely to be on each day of the year,

$$E[X]$$

$$= 365E[I_1]$$

$$= 365P[\text{day 1 is the birthday of at least one of the four people}]$$

which, by using complementation,

$$= 365\,[1 - P[\text{day 1 is not the birthday of any of the four people}]]$$

$$= 365\left[1 - \left(\frac{364}{365}\right)^4\right]$$

which, if calculated, is 3.98359....

---

13. Let $X_1, X_2, \ldots, X_n$ be independent random values drawn uniformly from $[0, 1]$. Let

$$A = \min\{X_1, X_2, \ldots, X_n\} \qquad \text{and} \qquad B = \max\{X_1, X_2, \ldots, X_n\}.$$

What is the probability that $A + B > 1$?

Solution: Let $\alpha = \Pr\{A + B > 1\}$ and $\beta = \Pr\{A + B < 1\}$. Note that $\alpha + \beta = 1$ because $\Pr\{A + B = 1\} = 0$.

Let $A' = \min\{1 - X_1, \ldots, 1 - X_n\}$ and $B' = \max\{1 - X_1, \ldots, 1 - X_n\}$. Since $1 - X_i$ is also uniform on $[0, 1]$ we have $\Pr\{A' + B' > 1\} = \alpha$ and $\Pr\{A' + B' < 1\} = \beta$.

Observe that $A' = 1 - B$ and $B' = 1 - A$. Therefore

$$\alpha = \Pr\{A' + B' > 1\} = \Pr\{(1 - A) + (1 - B) > 1\} = \Pr\{A + B < 1\} = \beta$$

hence $\alpha = \beta = \frac{1}{2}$.

14. Let X, Y and Z be random variables having mean 0 and unit variance. Show that

$$|\rho_{XZ} - \rho_{XY}\rho_{YZ}| \leq \sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{YZ}^2},$$

where ρ_{XZ} is the correlation of the random variables X and Z, similarly for ρ_{XY}, ρ_{YZ}. *Hint:* Write $XZ = [\rho_{XY}Y + (X - \rho_{XY}Y)][\rho_{YZ}Y + (Z - \rho_{YZ}Y)]$.

Solution: Note that $X - \rho_{XY}Y$ is uncorrelated with Y and has mean zero and variance $1 - \rho_{XY}^2$. Indeed,

$$E[(X - \rho_{XY}Y)Y] = E(XY) - \rho_{XY}E(Y^2) = \rho_{XY} - \rho_{XY} \cdot 1 = 0.$$

Mean zero is obvious as a linear combination of mean zero variables, while

$$E[(X - \rho_{XY}Y)^2] = E(X^2) - 2\rho_{XY}E(XY) + \rho_{XY}^2 E(Y^2) = 1 - 2\rho_{XY}^2 + \rho_{XY}^2 = 1 - \rho_{XY}^2.$$

Likewise, $Z - \rho_{YZ}Y$ is uncorrelated with Y and has mean zero and variance $1 - \rho_{YZ}^2$. Using the hint, we obtain from the vanishing correlations above that

$$\rho_{XZ} = E(XZ) = \rho_{XY}\rho_{YZ}E(Y^2) + E[(X - \rho_{XY}Y)(Z - \rho_{YZ}Y)]$$

and, from $E(Y^2) = 1$,

$$\rho_{XZ} - \rho_{XY}\rho_{YZ} = E[(X - \rho_{XY}Y)(Z - \rho_{YZ}Y)]$$

The Cauchy-Schartz inequality implies

$$\begin{aligned} |E[(X - \rho_{XY}Y)(Z - \rho_{YZ}Y)]| &\leq \sqrt{E[(X - \rho_{XY}Y)^2]E[(Z - \rho_{ZY}Y)^2]} \\ &= \sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{ZY}^2} \end{aligned}$$

---

15. We toss a fair coin until two consecutive heads or tails appear for the first time. Determine the probability that an even number of tosses will be required.

Solution: Let T be the number of tosses until either two consecutive heads or two consecutive tails appear for the first time. Therefore, the events

$$\begin{aligned} (T = 2) &= \{HH, TT\} \\ (T = 3) &= \{HTT, THH\} \\ (T = 4) &= \{HTHH, THTT\} \\ (T = 5) &= \{HTHTT, THTHH\}, \ldots etc. \end{aligned}$$

imply that for any integer $k \geq 2$, $P(T = k) = \frac{2}{2^k}$, and the probability that T is even is

$$\sum_{n=1}^{\infty} P(T = 2n) = \sum_{n=1}^{\infty} \frac{2}{2^{2n}} = 2 \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.$$
