On Synchronizing Sampling and Quantization for Stabilizing the Double Integrator under Binary Sensing

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Abstract—We revisit the problem of constructing finite state $\rho/\mu$ approximations for the purpose of certified-by-design control synthesis. We investigate in this context the problem of picking the ‘initial partition’ to enable successful control design for a benchmark problem, namely that of exponentially stabilizing the double integrator by switching between two available feedback gains under binary sensing constraints. We motivate the problem through two illustrative case studies, provide an analysis of the intuition gleaned from special instances of it, propose a general algorithm for choosing the initial partition taking into account the sampling frequency and available choices of feedback gains, and demonstrate the use of our algorithm in a set of illustrative examples.

I. INTRODUCTION

The past decade has witnessed increasing interest in using finite state approximate models of more complex dynamics, such as switching dynamics, for the purpose of certified-by-design control synthesis [1], [2], [3], [4], [5], [6]. In particular, a notion of finite state $\rho/\mu$ approximation was proposed in [7], explicitly identifying three properties that finite state approximations should satisfy to ensure that they are compatible with the objective of certified-by-design control synthesis for systems over finite alphabets. In parallel, a sequence of papers investigated various input-output [8], [9] and state-space based [10] approaches for constructing finite approximations with the desired $\rho/\mu$ approximation properties, thereby enabling systematic control design. In particular, the starting point of the state-space based approach [10] is a finite initial partition of the state-space. In practice, the choice of initial partition is critical to the success of the method, and the optimal choice of initial partition remains an open problem.

In this paper, we consider the problem of constructing finite state $\rho/\mu$ approximations for use in designing a stabilizing binary output switching controller for the double integrator. We have chosen the double integrator as a benchmark problem to study as it has been extensively used to model various interesting physical phenomena [11], [12], and is perhaps the simplest yet nontrivial instance of a switched system. Since we are interested in control under discrete sensing, we focus on the extreme case where binary sensors are used. Our solution approach builds on the approach proposed in [10], and focuses on the central question of choosing the initial partition so as to enable control design. Special cases of the problem considered in this paper were presented as illustrative examples in [14], [15]: For these special cases, a uniform initial partition was successfully used to construct an adequate $\rho/\mu$ approximation enabling systematic control design.

The contributions of the paper are as follows:

1) We illustrate, via a case study, the shortcomings of a uniform initial partition, thus motivating the need for a more sophisticated choice of initial partition in general.
2) We present an analysis of the system’s relevant dynamics and those of its finite state approximation, thereby developing the intuition for a more sophisticated choice of initial partition in the general setting.
3) We propose a general algorithm for constructing the initial partition, taking into account the sampling period and available feedback gains, and we show that it satisfies several desirable properties.
4) We demonstrate our algorithm via numerical examples.

Organization: We present the benchmark problem in Section II. We briefly survey the relevant background on $\rho/\mu$ approximations and the benchmark problem in Section III. We motivate the need for a more sophisticated choice of initial partition through case studies in Section IV. We present an analysis and develop the intuition to guide our choice in the general setting in Section V. We propose a new algorithm for choosing the initial partition and we establish its desirable properties in Section VI. We present illustrative examples in Section VII and directions for future work in Section VIII.

Notation: $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of nonnegative integers, reals and non-negative reals, respectively. Given $v \in \mathbb{R}^n$, $v'$ denotes its transpose and $||v|| = \sqrt{v'v}$ denotes its Euclidean norm. Given a function of time $x : \mathbb{R}_+ \rightarrow \mathbb{R}$, $x_0$ denotes its value at $t = 0$, that is $x_0 = x(0)$. Continuous time signals $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ will be denoted by $x(t)$ while sampled signals $x : \mathbb{Z}_+ \rightarrow \mathbb{R}$ will be denoted by $x[k]$: It is understood here that $x[k] = x(kT_s)$, where $T_s$ is the sampling period.

II. THE BENCHMARK PROBLEM

Consider the setup shown in Figure 1: Black lines represent analog systems and signals, while blue lines represent discrete systems and sampled signals, with sampling period...
The plant is a double integrator that can be placed in feedback with one of two static gains, $k_1$ and $k_2$. Switching between this pair of feedback gains is controlled by a switching controller: Control inputs $u_1$ and $u_2$ correspond to choosing $k_1$ and $k_2$, respectively, in the feedback loop. Binary sensor information $y[k]$ is available to the switching controller at the beginning of the $k^{th}$ sampling interval. At that point, the controller picks a choice of control input $u[k] \in \mathcal{U} = \{u_1, u_2\}$ and thus of feedback gain, and holds it until the beginning of the next sampling interval when the process is repeated.

Each of the closed-loop subsystems ($i \in \{1, 2\}$) is described by a state-space equation of the form

$$
\begin{aligned}
\dot{x} &= A_i x, \\
y &= \text{sign}(Cx),
\end{aligned}
$$

where $C = [c_1 \; c_2] \in \mathbb{R}^{1 \times 2}$ is given, and

$$
A_i = \begin{bmatrix} 0 & 1 \\ \frac{1}{k_i} & 0 \end{bmatrix}.
$$

The binary sensor distinguishes between two half-planes of the state-space, demarcated by a sensing line defined by its angle with the $x_1$ axis,

$$
\theta_s = \tan^{-1}(-\frac{c_1}{c_2}), \quad \theta_s \in [0, \pi).
$$

**Problem:** Given $k_1$, $k_2$, $C$ and sampling period $T_s$, design a switching controller that maximizes $R > 0$, such that

$$
\sup_{T \geq 0} \sum_{k=0}^{T} (v[k] + R) < \infty,
$$

where

$$
v[k] = \log(\frac{||x[k+1]||}{||x[k]||}).
$$

**Remark 1:** Performance objective (3) can be interpreted as provably ‘exponentially’ stabilizing the closed loop system, with $R$ corresponding to the provable rate of convergence. Indeed, by defining $k(x(0)) = 10^{S(x(0))-R}$ and $S(x(0)) = \sup_{T \geq 0} \sum_{i=0}^{T} (v(t) + R)$, (3) can be equivalently rewritten as

$$
||x(t)|| \leq k(x(0))10^{-R}||x(0)||, \quad \forall t, \ x(0) \in \mathbb{R}^2.
$$

This notion of convergence is thus slightly weaker than the standard exponential convergence since we do not require $k(x(0))$ to be uniformly bounded.

## III. Background & Preliminaries

We begin by briefly reviewing relevant recent work on certified-by-design control synthesis for systems over finite alphabets based on $\rho/\mu$ finite state approximations, which is the starting point of this paper. We then review an idealized version of the benchmark problem, highlighting the capabilities and limitations of switching control when the switching controller has access to the full state of the system in the absence of sampling. This allows us to establish limits on what can be achieved under sampling and binary sensing. Finally, we review the construction of $\rho/\mu$ approximations starting from finite initial partitions in this context.

### A. $\rho/\mu$ Approximations for Control Design

Given a switched system $P$ (i.e. control input $u(t) \in \mathcal{U}$, a finite control alphabet) with finite-valued sensors (i.e. sensor output $y(t) \in \mathcal{Y}$, a finite sensor alphabet) the basic idea is to construct a sequence of deterministic finite state machines $\{M_i\}_{i=1}^{\infty}$, chosen so that when perturbed by appropriate error systems $\{\Delta_i\}_{i=1}^{\infty}$ (Figure 2), they recover relevant aspects of the input/output behavior of the original plant $P$ while adequately approximating the performance objective. More precisely, the notion of $\rho/\mu$ approximation $[7]$ explicitly identified three properties that $\{M_i\}_{i=1}^{\infty}$, $\{\Delta_i\}_{i=1}^{\infty}$, and their interconnection should satisfy to enable certified-by-design controller synthesis. In this setting, a deterministic finite state machine (DFM) is understood to be a discrete-time system whose input and output alphabets and whose state set are all finite sets. The sequence $\{M_i\}_{i=1}^{\infty}$ is understood to represent a sequence of finite state approximations of the plant of increasing fidelity. The sequence $\{\Delta_i\}_{i=1}^{\infty}$ is understood to represent the corresponding sequence of approximation errors. The constructive approach proposed in $[8]$ specifically assigned to $\Delta_i$ the structure shown in Figure

![Diagram of Double Integrator with Switched Static Feedback and Binary Sensor](image-url)
3. For the special case of a binary alphabet, as in this paper.

\[ w = (y, \hat{y}) \text{ is defined to be 0 when } y \text{ and } \hat{y} \text{ match and 1 otherwise.} \]

One measure of the quality of approximation is the gain \( \gamma_i \) of the error system \( \Delta_i \), defined as the infimum of \( \gamma \) such that

\[
\inf_{T \geq 0} \sum_{t=0}^{T} \gamma \rho(u(t)) - \mu(w(t)) > -\infty \tag{4}
\]

is satisfied, where \( \rho : \mathcal{U} \rightarrow \mathbb{R}_+ \) is defined by \( \rho(u) = 1 \) and \( \mu : \mathcal{W} \rightarrow \mathbb{R}_+ \) is defined by \( \mu(w) = w \). Intuitively, \( \gamma_i \) thus represents the fraction of time that the outputs \( y \) and \( \hat{y} \) of \( P \) and \( \hat{M}_i \), respectively, disagree in the worst-case scenario. A key property of \( \rho/\mu \) approximations is the requirement that the gains of successive approximation errors in the sequence be monotonically non-increasing, that is \( \gamma_{i+1} \leq \gamma_i \).

![Fig. 3. Approximation error \( \Delta_i \) of \( P \) and \( \hat{M}_i \)](image)

Given a finite state approximation \( \hat{M}_i \) with the desired properties, with dynamics given by equations of the form

\[
q(t + 1) = f(q(t), u(t), y(t)) \\
\hat{y}(t) = g(q(t)) \\
\hat{v}(t) = h(q(t), u(t)) \tag{5}
\]

one attempts to design a full state feedback control law \( \phi \) such that the closed loop system consisting of \( \hat{M}_i \) in feedback with \( \phi \) satisfies an auxiliary performance objective, namely

\[
\inf_{T \geq 0} \sum_{t=0}^{T} \tau \mu(w(t)) - R - \hat{v}(q(t), \hat{v}(q(t))) - T \gamma_i \rho(u(t)) > -\infty \tag{6}
\]

for some \( \tau > 0, R > 0 \). If this problem admits a solution, \( \phi \) can be combined with \( \hat{M}_i \) to construct an (observer based) stabilizing controller for the original plant (Figure 4), guaranteeing exponential convergence at rate \( R \). The details of the approach can be found in [15]. On the other hand, if the problem does not admit a solution or the guaranteed rate is not satisfactory, one constructs the next approximation in the sequence, \( \hat{M}_{i+1} \), and repeats the process.

![Fig. 4. Internal structure of the finite state stabilizing controller \( K \)](image)

### B. The Double Integrator with Full State Feedback

The problem of designing stabilizing switching feedback control laws is fundamental in the study of switched systems [13]. While it remains largely open, several significant results have been established to date for classes of systems, including conic switching laws [16], [17], quadratic switching laws [18], [19], and sliding mode based control laws [20], [21], [22]. The switching controller is typically assumed to have full access to the states, or alternatively, to an output signal which is a linear function of the states. In practice, the ubiquity of discrete and coarse sensors [23] [24] justifies studying this fundamental problem in the presence of coarse, discrete sensing. The problem thus becomes more challenging due to state estimation, and its study has been limited so far.

In this Section, we review the relevant dynamics of the double integrator in feedback with static gains: Expressing the dynamics in polar coordinates give us better insight into the state trajectories of the system.

**Proposition 1:** The dynamics of system (1) in polar coordinates are given by:

\[
\dot{r}(t) = r(t)(k_i + 1)\sin \theta(t) \cos \theta(t), \\
\dot{\theta}(t) = k_i \cos^2 \theta(t) - \sin^2 \theta(t), \\
y(t) = \text{sign}(c_1 \cos \theta(t) + c_2 \sin \theta(t)).
\]

where \( r(t) = ||x(t)|| \) and \( \theta(t) = \tan^{-1}(\frac{y_2(t)}{y_1(t)}) \), \( i \in \{1, 2\} \).

**Proof:** By direct substitution, omitted for brevity.

Representative state trajectories for different qualitative values of the feedback gain \( k_i \) are sketched in Figure 5. Proposition 1 and Figure 5 highlight several features:

1) The evolution of \( \theta(t) \) is independent of \( r(t) \), for any choice of \( k_i \). Moreover, in the special case where \( k_i = -1 \), \( \dot{\theta}(t) = -1 \).

2) Likewise, the ratio \( \frac{\dot{r}(t)}{r(t)} \) is independent of the radial coordinate \( r(t) \), for any choice of \( k_i \).

3) For any \( k_i \neq 0 \), \( \dot{r}(t) < 0 \) in exactly two quadrants of the state space.

4) It is impossible to stabilize the system using two positive gains. Indeed, when the initial state is in quadrant I, the trajectory remains in quadrant I and \( \dot{r}(t) > 0, \forall t \).

5) Assuming full state feedback, it is always possible to stabilize the system using two gains of opposite signs.

6) Again assuming full state feedback, it is always possible to stabilize the system by using two distinct negative gains, one greater than or equal to -1 and the other less than or equal to it.

**Proposition 2:** When \( k_i < 0 \), the time \( T_R \) required for a state trajectory to rotate by \( \pi \) is given by

\[
T_R = \frac{\pi}{\sqrt{-k_i}} \tag{8}
\]

**Proof:** See Appendix.
C. Finite Approximations for the Benchmark Problem

In order to permit our readers to appreciate the contribution of this paper in an isolated context, we will briefly review the intuition behind the state-space based construction of $M_i$ for the plant of interest starting from a given choice of finite initial partition. Observations 1) and 2) in Section III-B indicate that the system state relevant to our problem effectively evolves on the unit circle. The finite initial partition $P_i = \{I_1, \ldots, I_{n_i}\}$ thus consists of a sequence of angle intervals, $I_j^i = [\alpha_j, \alpha_{j+1})$, with $\alpha_j < \alpha_{j+1}$ for $j = 1, 2, \ldots, n_i$ and $\alpha_1 + 2\pi = \alpha_{n_i+1}$.

The initial state $q_0$ of $M_i$ is then the union of all the elements of the initial partition, that is the entire unit circle. The set of states $Q_i$ of $M_i$ is associated with those elements in the power set of $P_i$ that can be reached from the initial state $q_0$ in a finite number of steps under some switching sequence. The details of the general construction can be found in [10]. In order to ensure the condition $\gamma_{i+1} \leq \gamma_i$, we require the sequence of initial partitions to be nested, meaning that every element $I_j^{i+1}$ of partition $P_{i+1}$ is entirely contained in some element $I_k^i$ of the $i^{th}$ partition $P_i$.

Beyond this requirement, the choice of initial partition was left open, to be chosen taking into account the underlying dynamics of the system. This is the central question addressed in this paper for the benchmark problem under consideration.

A final note here: When $M_i$ is subsequently used as a state observer in this setup, its state can be interpreted as a set valued estimate of the state of the plant.

IV. MOTIVATION

In this Section, we present two case studies that illustrate the limitations of a uniform initial partition in the general setting.

The first case study reviews the special case where one of the two gains equals $-1$, which was first reported in [15]. Specifically, let $k_1 = -1$, $k_2 = -3$ and $T_s = 0.2094$. The relevant data, using a uniform initial partition to construct $\{M_i\}$, is shown in Table II. Table I explains the parameters reported. It can be seen that control design succeeds in the first iteration. The DFM has only a few hundred states, and the guaranteed performance is already quite reasonable.

In the second case study, we attempt to use a uniform initial partition in a more general setting where neither of the two gains equals to $-1$. Specifically, for $k_1 = -0.5$, $k_2 = -3$ and $T_s = 0.2094$, we attempt to use the same choice of initial partition as in the first case study. The relevant data is shown in Table III. It can be seen that in spite of successive refinements of the initial partition leading to finite approximations with significantly more states, control design fails.

These two case studies highlight that instances where $k_1 = -1$ are special cases, and demonstrate that a uniform initial partition does not provide a workable solution for the more general setting. What is needed is a more sophisticated approach for choosing the initial partition, taking into account the sampling time and the available choices of gains. This is the focus of this paper.

V. INTUITION & ANALYSIS

In this Section, we first analyze the special case where $k_1 = -1$ which was addressed as an illustrative example in [15]. This gives us insight into the properties of a desirable initial partition. We then show how this intuition can be extended to the general setting.
A. Analysis of the Special Case: $k_1 = 1$

Simulations of this special case indicated a tradeoff between state estimation\(^1\) and meeting performance objectives: Specifically, stabilizing controllers periodically switch between $k_1 = -1$ to improve the state estimate and $k_2 = 1$ to force the state trajectory closer to the origin. Moreover, a uniform initial partition makes sense in view of the constant angular speed of the plant trajectories when $k_1 = -1$. To formalize this intuition, we examine the sampled system:

Proposition 3: Let $K_i = \sqrt{-k_i}$. The dynamics of (1), sampled with period $T_s$, are given in polar coordinates by:

$$r[n + 1] = \sqrt{(\cos^2(K_iT_s) + Z_1(\theta[n]) + Z_2(\theta[n]))r^2[n]},$$

$$\theta[n + 1] = \tan^{-1}\left(N(\theta[n])D(\theta[n])\right),$$

$$y[n] = \text{sign}(c_1 \cos(\theta[n]) + c_2 \sin(\theta[n])), \quad (9)$$

where

$$Z_1(\theta[n]) = \sin(2K_iT_s)(-K_i + \frac{1}{2K_i})\sin(\theta[n])\cos(\theta[n]),$$

$$Z_2(\theta[n]) = \sin^2(K_iT_s)(K_i^2\cos^2(\theta[n]) + \frac{1}{2K_i^2}\sin^2(\theta[n])),$$

$$N(\theta[n]) = -K_i\sin(K_iT_s)\cos(\theta[n]) + \cos(K_iT_s)\sin(\theta[n]),$$

$$D(\theta[n]) = \cos(K_iT_s)\cos(\theta[n]) + \frac{1}{2K_i}\sin(K_iT_s)\sin(\theta[n]).$$

Proof: The details are omitted for brevity. $\blacksquare$

Let $\Phi_i(\theta[n]) = \tan^{-1}\left(\frac{N(\theta[n])}{D(\theta[n])}\right)$, and introduce the following definition:

Definition 1: Consider the sampled plant $P$ as in (9), a $\rho/\mu$ approximation $M_i$ and a corresponding approximation error system $\Delta_i$ as shown in Figure 3. $\gamma_i(k_j)$ is the infimum of $\gamma$ such that $\Delta_i$ satisfies (4) when $u(t) = u_j$ for all $t \geq 0$.

Intuitively, $\gamma_i(k_j)$ reflects how well $M_i$ approximates the sampled double integrator with static gain feedback $k_j$.

Specifically, $\gamma_i(k_j) = 0$ indicates that when $u(t) = u_j$, all $M_i$ and $P$ exactly match, possibly after some finite transient.

Remark 2: $\gamma_i(k_j) \leq \gamma_i$, $\forall j \in \{1, 2\}$, since $\gamma_i$ corresponds to taking the infimum over an arbitrary choice of inputs, with $u(t)$ chosen to be identically equal to $u_j$ representing a particular choice.

Proposition 4: Consider system (9) and a uniform initial partition $P = \{I_1, \ldots, I_n\}$ with $\alpha_i = \theta_i$. Let $M$ and $\Delta$ be the corresponding finite state approximation and approximation error, respectively. If there exists integer $m \in \mathbb{Z}_+$ such that

$$nT_s = \frac{2m\pi}{\sqrt{-k_i}}, \quad (10)$$

holds for $k_i = -1$, then $\gamma_i\{-1\} = 0$.

Proof: It follows from (9) when $K_i = 1$ that

$$\Phi_i(\theta[n]) = \theta[n] - T_s$$

$$= \theta[n] - \frac{2m\pi}{n}$$

$$= \theta[n] - m\left(\frac{2\pi}{n}\right)$$

Noting that the length of each interval in this case is given by $2\pi/n$, we conclude that the image of each partition interval under map $\Phi_i$ is a single interval. In addition, starting from any state, the trajectory of the DFM associated with initial partition $P$ reaches a state corresponding to a single interval in a finite number of steps. Hence $\gamma_i(k_i) = 0$. $\blacksquare$

B. The General Setting: $k_1 \neq -1$, $k_2 \neq 1$

We first consider the case where we have freedom to choose $T_s$ to satisfy sampling constrain (10). Then we extend the intuition to more general case where $T_s$ is given.

Definition 2: Given an initial partition $P = \{I_1, \ldots, I_n\}$, we say interval $I_m = [\alpha_m, \alpha_{m+1})$ is matched to gain $k_i$ if there exists $j, l \neq m$ such that $\alpha_j = \Phi_i(\alpha_m)$ and $\alpha_l = \Phi_i(\alpha_{m+1})$.

Proposition 5: Consider system (9). Let $k_i < 0$ and assume there exists an $n \in \mathbb{Z}_+$ such that (10) is satisfied with $m = 1$. Then there exists an initial partition $P$, a corresponding finite state machine approximation $M$ and approximation error $\Delta$ such that $\gamma\{k_i\} = 0$.

Proof: The proof is by construction, start from $\theta(0) = \theta_s$, it takes $2mn$ steps for the trajectory to rotate $360^\circ$ and assume the trajectory sequence to be $\{\theta(t)\}_{t=0}^{2mn}$, then define $\alpha_{t+1} = \theta(t)$ for each step $t$, $0 < t \leq 2mn$. From simple computation it can be seen that $\alpha_{m+1} + \alpha_{m+1} = \alpha_1 + \pi$, and $\alpha_{2m+1} = \alpha_1 + 2\pi$.

By this construction, all the intervals are matched to dynamic of $k$. Hence follows the proof of Proposition 4, the conclusion holds. $\blacksquare$

However, in the setting under consideration, $T_s$ is given. As such, we generally cannot ensure that sampling constraint (10) holds. Fortunately, we can still design a reasonably good initial partition.

Proposition 6: Consider system (9). Let $k_i < 0$. There exists an initial partition $P$, a corresponding finite state machine approximation $M$ and approximation error $\Delta$ such that $\gamma\{k_i\} \leq \frac{1}{n_p}$, where $n_p = \text{floor}(\frac{n}{\sqrt{-k_i}T_s})$, $n_p \geq 2$.

Fig. 6. Proposed Initial Partition
Proof: We need only consider the case when (10) is not satisfied. We can define the angles of \(1\) to \(np+1\), following the construction in Proposition 5 and define \(\alpha_{np+2} = \theta_s - \pi\). Thus, we have picked the angles in the right half plane of the sensing line, all intervals are matched to dynamics of \(k\) except the interval \([\alpha_{np+1}, \alpha_{np+2}]\) signed by the red region as shown in Figure 6. For the remaining angles, define \(\alpha_{np+1+j} = \alpha_j - \pi, 1 \leq j \leq np\).

To see why the \(\gamma(k) \leq \frac{1}{np}\), note that by construction, the state converges to a one-interval state, say \(q_s\), after a finite number of transitions. Start from \(q_s\) and follow the state trajectory: It takes at least \(2np\) steps to transit once again to \(q_s\). Notice that, the state trajectory from \(q_s\) to \(q_s\) is a cycle and there are at most two states in this cycle that \(w\) might equal to \(1\), hence \(\gamma \leq \frac{1}{np}\).

The proof of Proposition 6 outlines a constructive algorithm to design a good initial partition such that the intervals are matched to one of the two gains.

VI. PROPOSED ALGORITHM & ANALYSIS

A. Proposed Algorithm

Algorithm 1: Constructive Algorithm

Given: Feedback gains \(k_1, k_2\), sampling time \(T_s\), sensing angle \(\theta_s\).

1. if Two negative gains are used then
   1. choose \(k_1 \in (-1, 0)\) and let \(k_2 = \sqrt{-k_1}\)
   2. if Opposite sign gains are used then
      1. choose \(k_1 \in (-\infty, 0)\) and let \(k_2 = \sqrt{-k_1}\)
   3. Initialization: Set \(i = 1\)
   4. while Control synthesis fails or performance is not good enough do
   5. Compute \(m = q^{i-1}\);
   6. Compute \(T_j^i = \frac{T_s}{m}\);
   7. Compute \(n_{i,j}^m = \text{floor}(\frac{\pi}{T_j^i})\);
   8. Pick the initial partition \(\{\alpha_{j}^i\}_{j=1}^{2n_{i,j}^m+3}\), according to:
      \[
      \alpha_j^i = \tan^{-1}\left(-\frac{k_s \sin(k_s T_j^i) \cos \theta_s + \cos(k_s T_j^i) \sin \theta_s}{\cos(k_s T_j^i) \cos \theta_s + \frac{k_s}{T_j^i} \sin(k_s T_j^i) \sin \theta_s}\right)
      \]
      for \(j = 1, 2, \ldots, n_{i,j}^m + 1\)
      \[
      \alpha_{(n_{i,j}^m+2)}^i = \theta_s - \pi
      \]
      \[
      \alpha_{(j+n_{i,j}^m+2)}^i = \alpha_j^i - \pi
      \]
      for \(j = 1, 2, \ldots, n_{i,j}^m + 1\)
      \[
      \alpha_{j}^i = \alpha_{2n_{i,j}^m+4-j}^i + 2\pi
      \]
      for \(j = 1, 2, \ldots, 2n_{i,j}^m + 3\)
   9. Construct \(\tilde{M}_i\) and attempt control synthesis;
   10. Set \(i = i+1\) and go back to step 6;

B. Analysis of The Algorithm

In this Section, we check whether the proposed initial partition satisfies the following two properties:

(i) The approximation quality is improving after each iteration.
(ii) When two negative gains are used, the resulting approximation model is self-transition free.

Proposition 7: Algorithm 1 generates a sequence of nested initial partitions.

Proof: Consider the \(i\)th iteration, and let \(\{\alpha_1^i, \alpha_2^i, \ldots, \alpha_{2n_{i,j}^m+3}^i\}\) be the set of angles defining the initial partition. There are two situations:

1. The angles that are directed defined: \(\alpha_1 = \alpha_{2n_{i,j}^m+3} = \theta_s\), \(\alpha_{n_{i,j}^m+2} = \theta_s - \pi\),
2. The angles that are reached from \(\theta_s\): \(\alpha_2, \ldots, \alpha_{n_{i,j}^m+1}\) and those reached from \(\theta_s + \pi\): \(\alpha_{n_{i,j}^m+3}, \ldots, \alpha_{2n_{i,j}^m+2}\).

Now consider the \((i+1)\)th iteration, the angles in situation (1) are included in the new partition, and for situation (2), since \(T_{i+1}^* = 2T_{i}^* + 1\), therefore these angles are also included in the new partition.

Remark 3: In principle, when two negative gains are used we can choose either \(k_1 \in (-\infty, -1)\) or \(k_1 \in (-1, 0)\) as the basis for the initial partition and end up with successful design. However in practice, using \(k_1 \in (-\infty, -1)\) appears to require larger memory and consequently higher computational cost, as will be illustrated in the next Section. A concrete analysis of this observation remains a current subject of study.

VII. ILLUSTRATIVE EXAMPLES

We present two illustrative examples to demonstrate the new algorithm proposed in this paper.

Example 1: Let \(C = [1, 0]\), \(T_s = 0.2\) and assume that two negative gains are used, namely \(-0.5\) and \(-4\). We present two simulations for comparison: Specifically in the first we base the initial partition on \(k_1 = -0.5\), and in the second we base the initial partition on \(k_1 = -4\). The relevant data is shown in Tables IV and V.

| Example 1: Initial partition based on \(k_1 = -0.5\) |
|---|---|---|---|
| \(i\) | \(n_1\) | \(N_1\) | \(g_1\) | \(R\) | \(p\) |
| 1 | 46 | 421 | 0.7059 | 0.0072 | 44 |
| 2 | 90 | 899 | 0.6471 | 0.0151 | 43 |
| 3 | 178 | 2999 | 0.5625 | 0.0161 | 55 |

Comparing the two tables, we see that from both a performance and a memory perspective, using \(k_1 = -0.5\) as the basis for initial partition is indeed a better choice. As expected, in both cases the provable rate of convergence
Table V

Example 1: Initial partition based on $k_1 = -4$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$n_i$</th>
<th>$N_i$</th>
<th>$\gamma_i$</th>
<th>$R$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>529</td>
<td>1</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
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<td>0.5417</td>
<td>0.0017</td>
<td>114</td>
</tr>
<tr>
<td>3</td>
<td>252</td>
<td>5511</td>
<td>0.4667</td>
<td>0.0090</td>
<td>50</td>
</tr>
</tbody>
</table>

Increases and the gain bound for the approximation error decreases with increasing $i$ and hence finer initial partition.

A representative state trajectory start from initial point $[0, 1]'$ is shown in Figure 7 for the case where $i = 1$. The state trajectory of the original system is plotted on the left, while a plot of the trajectory’s Euclidean distance from the origin versus time is shown on the right.

![Fig. 7. Simulation Result for $i=1$](image)

The comparison of the implementation results of three iterations is shown in Figure 8. The black line represent the ideal case (full state feedback, sampled case), while blue, green, and red represent the results of $i = 1$, $i = 2$ and $i = 3$, respectively.

![Fig. 8. Comparison of Different Value of $n$ for Example 3](image)

Example 2: In this example, we assume $C = [1, 0]'$, the sampling time $T_s = 0.1$ and two gains with opposite signs are used, specifically, $k_1 = -2$, $k_2 = 1$. The relevant data for three iterations are shown in Table VI.

In this case, the provable approximation gain $\gamma$ remain unchanged and the provable rate of convergence increases at each iteration. As before, a representative state trajectory start from initial point $[1, -1]'$ is shown in Figure 9 for the case where $i = 1$.

![Fig. 9. Simulation Result for $i=1$](image)

The comparison of the implementation results of three iterations are shown in Figure 10. The black line represent the ideal case (full state feedback, sampled case), while blue, green, and red represent the results of $i = 1$, $i = 2$ and $i = 3$ respectively.

![Fig. 10. Comparison of Different Value of $n$ for Example 2](image)

VIII. Future Work

Our future work will be in two directions: First, we will investigate the optimality of the proposed algorithm. Second, we will attempt to establish bounds on the sizes of the $\rho/\mu$ approximations required for successful design, or more generally, for achieving a desired guaranteed rate of convergence.

Table VI

Data for Example 2

<table>
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<tr>
<th>$i$</th>
<th>$n_i$</th>
<th>$N_i$</th>
<th>$\gamma_i$</th>
<th>$R$</th>
<th>$p$</th>
</tr>
</thead>
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</table>
IX. ACKNOWLEDGMENTS
This research was supported by AFOSR YIP award FA9550-11-1-0118 and NSF CAREER award ECCS 0954601.

APPENDIX

Proof of Proposition 2: Let \( x_{10} = x_1(0) \) and \( x_{20} = x_2(0) \) be the coordinates of the initial state \( x_0(0) \), and let \( \theta_0 = \tan^{-1}(\frac{x_{20}}{x_{10}}) \). When the trajectory rotates by 180°, we have \( \tan(\theta(t)) = \tan(\theta_0 + \pi) \), where \( \tan(\theta_0 + \pi) = \frac{x_1(t)}{x_2(t)} \). We thus have:

\[
x_2(t)x_{10} = x_1(t)x_{20}.
\]

Solving (1) with \( e^{\sqrt{k_i}t} \) written in trigonometric form, we have:

\[
x_1(t) = \cos(\sqrt{-k_i}t)x_1(0) + \frac{1}{\sqrt{-k_i}}\sin(\sqrt{-k_i}t)x_2(0)
\]

\[
x_2(t) = -\sqrt{-k_i}\sin(\sqrt{-k_i}t)x_1(0) + \cos(\sqrt{-k_i}t)x_2(0)
\]

Substituting these two equations, we get:

\[
\sin(\sqrt{-k_i}t)(k_1x_{10}^2 - x_{20}^2) = 0.
\]

Since \( k_1 < 0 \), \( (k_1x_{10}^2 - x_{20}^2) < 0 \) whenever \( x(0) \neq 0 \), and (8) is the solution.

REFERENCES


