On Subspace Decompositions of Finite Horizon Dynamic Programming Problems

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Abstract—We consider finite horizon dynamic programming problems where the dynamics are linear over a finite dimensional state-space and where the cost function depends only on the state. Starting from a decomposition of the state-space under the relevant linear transformation, we derive conditions under which the dynamic programming problem decomposes into a set of smaller problems that can be solved independently, with their combined solutions yielding the optimal solution of the original problem. For linear systems over finite fields, we show that the resulting reduction in complexity is exponential.

I. INTRODUCTION

Dynamic programming (DP), pioneered by Bellman [1], has found wide-ranging applications in diverse areas including operations research [2], bioinformatics [9], biology [8] and coding theory and cryptography [4]. The well-known Principle of Optimality, which results in a fairly intuitive solution approach, suffers from the curse of dimensionality: The computational complexity of the DP algorithm increases exponentially with the dimensions of the underlying state-space and the input space. Exceptionally, the well-studied case of Euclidean state-space, linear dynamics, and quadratic cost function is known to admit an elegant closed form solution in the infinite horizon case [2].

In our past research efforts, we explored finite state machines [17], [15] and their use as approximate models of more complex hybrid systems [13] for the purpose of certified-by-design control synthesis [14], [16]. To date, our study has not attempted to take advantage of algebraic structure one may impose on these finite memory models. In the present paper, we thus report on our initial results showing that the resulting reduction in complexity is exponential.

II. PROBLEM SETUP

A. Preliminaries

Let $X$ and $U$ be finite dimensional vector spaces defined over a common field $F$, with $\dim(X) = n$ and $\dim(U) = m$. Consider the discrete-time dynamical system $f$ defined by state transition equation

$$x_{t+1} = f(x_t, u_t)$$

from a decomposition of the state-space under the relevant linear transformation, we derive necessary and sufficient conditions for the dynamic programming problem to decompose into a set of smaller problems, and we formulate a readily verifiable sufficient condition for such a decomposition to exist. Finally, we explicitly characterize the resulting reduction in computational complexity for the finite memory setting.

It should be noted that decompositions of the Linear Quadratic dynamic programming problem, as well as related problems, have been considered in a different context, namely in decentralized control setups where the feedback control law needs to satisfy prescribed structural constraints consistent with imposed information structures [10], [5], [7], [12]. In contrast, our results are not motivated by particular information structures, but rather by the desire to exploit the algebraic structure of the underlying dynamics in order to reduce the complexity of solving the problem.

Organization: We review finite-horizon dynamic programming in Section II, and we formulate the problem of interest to us. We present the proof of the main decomposition result in Section IV following a set of intermediate results derived in Section III, and we show that this decomposition leads to an exponential reduction for vector spaces defined over finite fields in Section V. We present illustrative examples in Section VI and directions for future research in Section VII.

Notation: $\mathbb{R}^+$ denotes the set of non-negative reals. For sets $X$ and $Y$, $X^Y$ denotes the set of all maps from $Y$ to $X$ while $id_X$ denotes the identity map on $X$. For $f : X \rightarrow \mathbb{R}^+$, $\arg\min f(x)$ denotes the set of arguments in $X$ that minimize $f$. For maps $f : X \rightarrow Y$ and $g : Y \rightarrow W$, $g \circ f$ denotes the composite map from $X$ to $W$ defined by $g \circ f(x) = g(f(x))$. For a vector space $X$ over a field $F$, $x \in X$, $\dim(X)$ and $\langle x \rangle > 0$ denote the dimension of $X$ and the subspace spanned by $x$ over $F$, respectively. $R(B)$ and $N(B)$ denote the range space and null space, respectively, of a linear operator $B$, and $\oplus$ denotes the direct sum of subspaces. For $\sigma : X \rightarrow A$, $X = S \oplus V$, $\sigma|_S$ denotes the map from $S$ to $A$ defined by $\sigma|_S(x) = \sigma(x)$, for all $x \in S$. If $H$ is a matrix, we denote by $H^T$ its transpose.
where \( x_t \in \mathcal{X}, u_t \in \mathcal{U}, f : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \) is given, \( t \in T = \{0, 1, \ldots, T-1\} \) and \( T \) is a given positive integer. Consider also a non-negative cost function of the state,

\[
g : \mathcal{X} \to \mathbb{R}^+. \tag{2}
\]

Given any initial state \( x_0 \in \mathcal{X} \), we wish to compute the optimal policy \( \pi^*(x_0) \in \mathcal{U}^T \) that minimizes the additive cost

\[
J(x_0, \pi) = \sum_{t=0}^{T} g(x_t)
\]

over all policies \( \pi \in \mathcal{U}^T \).

The solution of this finite horizon Dynamic Programming (DP) problem hinges on the Principle of Optimality [1]. Let \( T_t = \{t, t+1, \ldots, T-1\} \) and define the cost-to-go function

\[
J_t^*(x) = \min_{u \in \mathcal{U}} \sum_{t=0}^{T} g(x_{r,t}) , \quad t \in T
\]

for the system evolving according to (1) with \( x_t = x \). The principle of optimality can then be stated as

\[
J_t^*(x) = g(x) + \min_{u \in \mathcal{U}} J_{t+1}^*(f(x, u)) \tag{3}
\]

for any \( t \in T \) and \( x \in \mathcal{X} \). This observation is used as the basis of the recursive DP algorithm, which is solved backwards in time [2] for the optimal cost-to-go function and a corresponding optimal controller \( u^* : \mathcal{X} \times T \to \mathcal{U} \) satisfying

\[
u^*(x, t) \in \arg \min_{u \in \mathcal{U}} J_{t+1}^*(f(x, u_t)).
\]

For notational simplicity, \( u^*_t(x) \) will denote \( u^*(x, t) \).

Note that while the dynamic programming problem is typically formulated in a setting where \( \mathcal{X} = \mathbb{R}^n \) and \( \mathcal{U} = \mathbb{R}^m \), it is straightforward to verify that the Principle of Optimality and the iterative DP algorithm still hold, with no modifications, in the more general setting considered here.

### B. Problem Statement

In this paper, we are specifically interested in the case where the underlying dynamics of the system are linear. That is, state transition equation (1), can be written as

\[
x_{t+1} = Ax_t + Bu_t \tag{4}
\]

where \( A : \mathcal{X} \to \mathcal{X} \) and \( B : \mathcal{U} \to \mathcal{X} \) are given linear operators. We will use \( DP(A, B, g) \) to denote the finite horizon dynamic programming problem associated with linear dynamics (4) and cost function (2).

Now assume that \( \mathcal{X} \) can be decomposed into the direct sum of \( A \)-invariant subspaces, that is

\[
\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_r
\]

where \( A\mathcal{X}_i \subseteq \mathcal{X}_i, i \in \mathcal{I} = \{1, \ldots, r\}, r > 1 \). We will refer to such a decomposition as a decomposition of \( \mathcal{X} \) over \( A \).

**Remark 1:** Such a decomposition arises naturally regardless of the underlying field \( \mathcal{F} \). Indeed, in the standard setting where \( \mathcal{X} = \mathbb{R}^n \) and the eigenvalues of \( A \) are real, this decomposition is related to the Jordan canonical form of matrix \( A \). The more general case is addressed in the literature (see Chapter 7 in [11] and Chapter XIV.2 in [6]).

Consider the subspaces \( \mathcal{E}_i \) of the input space \( \mathcal{U} \) defined by

\[
\mathcal{E}_i = \{ u \in \mathcal{U} | Bu \in \mathcal{X}_i \}, \quad \forall i \in \mathcal{I}
\]

and define \( r \) linear subsystems with dynamics \( (A_i, B_i, g_i) \) given by \( A_i = A|\mathcal{X}_i, B_i = B|\mathcal{E}_i, \forall i \in \mathcal{I} \). By defining additionally \( g_i : \mathcal{X}_i \to \mathbb{R}^+ \) by \( g_i = g|\mathcal{X}_i, \forall i \in \mathcal{I} \), we can pose \( r \) dynamic programming problems \( DP(A_i, B_i, g_i), i \in \mathcal{I} \). Denoting the cost-to-go function and a corresponding optimal controller of the \( i \)th problem by \( J^*_{i,t}, u^*_{i,t} \), respectively, we propose the following definition:

**Definition 1:** The family of problems \( DP(A_i, B_i, g_i), i \in \mathcal{I} \), is said to be a decomposition of the problem \( DP(A, B, g) \) if the following two conditions hold:

1. For every \( x \in \mathcal{X}, x = x_1 + \cdots + x_r \), and for every \( t \in T \), there exists a
   \[
u^*_t(x) \in \arg \min_{u \in \mathcal{U}} J_{t+1}^*(Ax + Bu)
   \]
   and a family
   \[
u^*_t(x_i) \in \arg \min_{u \in \mathcal{U}} J_{t+1}^*(A_i x + B_i u), \quad i \in \mathcal{I}
   \]
   such that \( u^*_t(x) = K(u^*_{1,t}(x_1), \ldots, u^*_{r,t}(x_r)) \) for some function \( K \) that is independent of the specific instance of the problem, the choice of \( x \) and the choice of \( t \).

2. For every \( t \in \{0, \ldots, T\} \), we have
   \[
   J^*_t = H(J^*_{1,t}, \ldots, J^*_{r,t})
   \]
   for some function \( H \) that is independent of the specific instance of the problem, the choice of \( x \) and the choice of \( t \).

In particular, when \( K(u^*_{1,t}(x_1), \ldots, u^*_{r,t}(x_r)) = u^*_{1,t}(x_1) + \cdots + u^*_{r,t}(x_r) \) and \( H(J^*_{1,t}(x_1), \ldots, J^*_{r,t}(x_r)) = J^*_{1,t}(x_1) + \cdots + J^*_{r,t}(x_r) \), the decomposition will be referred to as additive.

We are now ready to state the two problems of interest to us:

**Problem 1:** Derive necessary and sufficient conditions to ensure that the family \( DP(A_i, B_i, g_i), i \in \mathcal{I} \), is an additive decomposition of \( DP(A, B, g) \).

**Problem 2:** When these conditions are met, explicitly characterize the resulting reduction in computational complexity for the finite memory setting of interest.

### III. Relevant Notions and Intermediate Results

**Definition 2:** Let \( A : \mathcal{X} \to \mathcal{X} \) be a linear operator on vector space \( \mathcal{X} \), and consider a decomposition \( \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_r \) of \( \mathcal{X} \) over \( A \). \( g : \mathcal{X} \to \mathbb{R}^+ \) is said to split over \( \mathcal{X} \) if

\[
g(x) = g(x_1) + \cdots + g(x_r)
\]

In general, we have that \( r \geq 1 \).
holds for all \( x \in \mathcal{X} \), where \( x = x_1 + \ldots + x_r, x_i \in \mathcal{X}_i \), is the unique decomposition of \( x \).

We will use \( \mathcal{G}_s \) to denote (with a slight abuse in notation) the set of all functions that split over a given decomposition of \( \mathcal{X} \) over \( A \). Note that every element of \( \mathcal{G}_s \) necessarily maps \( 0 \) to \( 0 \):

**Lemma 1:** Let \( g \in \mathcal{G}_s \). Then \( g(0) = 0 \).

**Proof:** \( 0 \in \mathcal{X}_i \cap \mathcal{X}_j \Rightarrow g(0) = g(0+0) = g(0)+g(0) \Rightarrow g(0) = 0 \).

The following observation will be helpful in the proof of the main results:

**Lemma 2:** Consider dynamics (4), let \( \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_r \) be a decomposition of \( \mathcal{X} \) over \( A \), and let \( g \in \mathcal{G}_s \). Then for any \( x_i \in \mathcal{X}_i \), min \( g(Ax_i + Bu_i) = \min_{u \in \mathcal{E}i} g(Ax_i + Bu_i) \).

**Proof:** \( \begin{align*}
g(x_i) & = \min_{u \in \mathcal{E}i} g(Ax_i + Bu_i) \\
g(\sum_{i \in \mathcal{I}} g(x_i)) & = \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} \sum_{i \in \mathcal{I}} \left[ g(Ax_i + Bu_i) \right] \\
& = \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} \left[ \sum_{i \in \mathcal{I}} g(Ax_i + Bu_i) \right] \\
& = \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} \sum_{i \in \mathcal{I}} \left[ g(Ax_i + Bu_i) \right]
\end{align*} \)

since we can select \( u_j = 0 \) for \( j \neq i \).

**IV. Subspace Decomposition of the DP Problem**

We begin by deriving a necessary and sufficient condition for a given DP problem to admit an additve decomposition.

**Lemma 3:** Consider a DP problem \( \{DP(A_i, B_i, g_i) \}_{i \in \mathcal{I}} \) is an additve decomposition of \( DP(A, B, g) \) defined over time horizon \( T \), and let \( \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_r \) be a decomposition of \( \mathcal{X} \) over \( A \). The following three statements are equivalent:

1) The family of DP problems \( \{DP(A_i, B_i, g_i) \}_{i \in \mathcal{I}} \) is an additive decomposition of \( DP(A, B, g) \).
2) a) \( g \in \mathcal{G}_s \)
   b) \( \arg\min_{u \in \mathcal{U}_i} J_{t+1}^* (Ax_i + Bu_i) \cap \sum_{i \in \mathcal{I}} \mathcal{E}_i \neq 0, \ \forall x \in \mathcal{X}, \ \forall t \in \mathcal{T} \)
3) a) \( J_t^* \in \mathcal{G}_s, \ \forall t \in \mathcal{T} \)
   b) \( \arg\min_{u \in \mathcal{U}_i} J_{t+1}^* (Ax_i + Bu_i) \cap \sum_{i \in \mathcal{I}} \arg\min_{u \in \mathcal{U}_i} J_{t+1}^* (Ax_i + Bu_i) \neq 0, \ \forall x \in \mathcal{X}, \ \forall t \in \mathcal{T}, \ \text{where } x_i \text{ is the unique component of } x \text{ in } \mathcal{X}_i \)

**Proof:** (2 \( \Rightarrow \) 3) We start by showing using backward induction on \( t \) in \( \mathcal{T} \) that \( J_t^* \in \mathcal{G}_s \). For \( t = T \) we have by definition \( J_T^* = g \Rightarrow J_T^* \in \mathcal{G}_s \). Let \( x \in \mathcal{X} \) with \( x_i \) denoting its unique component in \( \mathcal{X}_i \). Then \( J_{T-1}^* (x) = \)

\( \begin{align*}
g(x) & = \min_{u \in \mathcal{U}} J_T^* (Ax + Bu) \\
g(x) & \equiv g(x) + \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} J_T^* (Ax + Bu) \\
& = \sum_{i \in \mathcal{I}} g(x_i) + \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} \sum_{i \in \mathcal{I}} J_T^* (Ax_i + Bu_i)
\end{align*} \)

where \( \text{hyp} \) stands for hypothesis and \( \text{p.o} \) stands for principle of optimality. The inductive step now can be performed in exactly the same manner as above to show that \( J_{t+1}^* \in \mathcal{G}_s \Rightarrow J_t^* \in \mathcal{G}_s \), for arbitrary \( t \in \mathcal{T} \).

We now prove by backward induction on \( t \in \mathcal{T} \) that \( J_t^* \mid x_i = J_t^* (x_i) \mid x_i \).

**Lemma 2:** Let \( \mathcal{X}_1 \oplus \ldots \oplus \mathcal{X}_r \) be a decomposition of \( \mathcal{X} \) over \( A \). Note that every element of \( \mathcal{G}_s \) necessarily maps \( 0 \) to \( 0 \):

**Proof:** \( (2 \Rightarrow 3) \) We start by showing using backward induction on \( t \in \mathcal{T} \) that \( J_t^* \in \mathcal{G}_s \). For \( t = T \) we have by definition \( J_T^* = g \Rightarrow J_T^* \in \mathcal{G}_s \). Let \( x \in \mathcal{X} \) with \( x_i \) denoting its unique component in \( \mathcal{X}_i \). Then \( J_{T-1}^* (x) = \)

\( \begin{align*}
g(x) & = \min_{u \in \mathcal{U}} J_T^* (Ax + Bu) \\
g(x) & \equiv g(x) + \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} J_T^* (Ax + Bu) \\
& = \sum_{i \in \mathcal{I}} g(x_i) + \min_{u \in \sum_{i \in \mathcal{I}} \mathcal{E}_i} \sum_{i \in \mathcal{I}} J_T^* (Ax_i + Bu_i)
\end{align*} \)

where \( \text{hyp} \) stands for hypothesis and \( \text{p.o} \) stands for principle of optimality. The inductive step now can be performed in exactly the same manner as above to show that \( J_{t+1}^* \in \mathcal{G}_s \Rightarrow J_t^* \in \mathcal{G}_s \), for arbitrary \( t \in \mathcal{T} \).
with $\sum_{i \in I} \arg \min_{u_i \in \mathcal{E}_i} J^*_{i,t+1}(A_i x_i + B_i u_i) \subseteq \sum_{i \in I} \mathcal{E}_i$ imply $\arg \min_{u_i \in \mathcal{E}_i} J^*_{i,t+1}(A x + B u) \cap \sum_{i \in I} \mathcal{E}_i \neq \emptyset$. (1) $\Rightarrow$ (3) We only need to show that $J^*_i \in \mathcal{G}_i$. By hypothesis $J^*_i(x) = \sum_{i \in I} J^*_{i,t}(x_i)$, $\forall t \in T$. In particular, for $t = T$ this relation yields $g(x) = \sum_{i \in I} g(x_i)$, i.e., $g \in \mathcal{G}_s$ and by Lemma 1 $g(0) = 0$. Then, it is seen by induction on $t \in T$ that $J^*_i(0) = 0, \forall t \in T$, which in turn implies that $J^*_i(0) = 0, \forall t \in T, \forall i \in I$. Setting $x = x_i$ in $J^*_i(x) = \sum_{i \in I} J^*_{i,t}(x_i)$ we get $J^*_i(x_i) = J^*_{i,t}(x_i)$, thus $J^*_i(x) = \sum_{i \in I} J^*_{i,t}(x_i), \forall t \in T$, i.e., $J^*_i \in \mathcal{G}_s, \forall t \in T$. (3) $\Rightarrow$ (1) We only need to show that $J^*_i(x) = \sum_{i \in I} J^*_{i,t}(x_i)$, where $x_i$ is the unique component of $x$ in $\mathcal{X}_i$. But this follows directly from the proof of (2) $\Rightarrow$ (3) whence $J^*_i | \mathcal{X}_i = J^*_{i,t}$, combined with the hypothesis $J^*_i \in \mathcal{G}_s$.

The above result is mainly of theoretical relevance, since checking condition 2(b) effectively requires solving the original DP problem. However Lemma 3 has an important followup:

**Theorem 1**: If $g \in \mathcal{G}_s$ and

$$\mathcal{R}(B) = \oplus_{i \in I} [\mathcal{R}(B) \cap \mathcal{X}_i],$$

(5)

then the family of problems $\{DP(A_i, B_i, g_i)\}_{i \in I}$ is an additive decomposition of $DP(A, B, g)$.

**Proof**: We will show that condition (5) $\Rightarrow$ $U = \sum_{i \in I} \mathcal{E}_i$.

Take $u \in \mathcal{U}$. Then $Bu \in \mathcal{R}(B) \Rightarrow Bu = \sum_{i \in I} b_i = \sum_{i \in I} \mathcal{E}_i$. Hence there exists $u_i \in \mathcal{E}_i$ such that $b_i = Bu_i \Rightarrow Bu = B(\sum_{i \in I} u_i) \Rightarrow u = \sum_{i \in I} u_i \in \mathcal{N}(B)$. Since $\mathcal{N}(B) \subseteq \mathcal{E}_i, \forall i \in I$, we have that $u \in \sum_{i \in I} \mathcal{E}_i$ and so $U = \sum_{i \in I} \mathcal{E}_i$. Thus condition 2(b) of Lemma 3 holds, and the present Theorem follows.

Notice also that the validity of condition (5) can be readily verified by inspecting the original problem. Intuitively, Thorem 1 states that if the subspaces are independently affected by the inputs and the cost function splits, then the DP problem decomposes. Note that while this condition is sufficient, it is not necessary as it is indeed possible to satisfy condition 2(b) of Lemma 3 when $U \neq \sum_{i \in I} \mathcal{E}_i$. Readers are referred to Section VI for a relevant illustrative example.

**V. Complexity Reduction: Linear Systems over Finite Fields**

In this section, building on our results, we characterize the reduction in the complexity of solving the finite-horizon DP problem resulting from the proposed subspace decomposition for the finite memory case where $\mathcal{F}$ is a finite field of characteristic $p$, where $p$ is prime.

Let $g$ be the cardinality of $\mathcal{F}$: State-space $\mathcal{X}$ and input space $\mathcal{U}$ consist of precisely $q^p$ and $q^m$ elements respectively. For arbitrary $g : \mathcal{X} \to \mathbb{R}^+$, an optimal control $u^*$ can be found by the principle of optimality (3) via direct computation of the quantity $\min_{u \in \mathcal{U}} J^*_{i,t+1}(A x + B u)$. This process requires for $t = T - 1, T - 2, \ldots, 0$ and for every $x \in \mathcal{X}$ evaluation of the quantity $J^*_{i,t+1}(A x + B u)$ for all $u \in \mathcal{U}$. Thus, the computational complexity of the problem consists of $T q^{n+m}$ function evaluations.

However, if the DP problem decomposes as in Lemma 3, we have in general an exponential reduction in the complexity:

**Corollary 1**: Consider the DP problem $DP(A, B, g)$ over time horizon $T$, let $g$ be the cardinality of the underlying field $\mathcal{F}$, let $\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_r$ be a decomposition of $\mathcal{X}$ over $\mathcal{A}$, and assume that the family of problems $DP(A_i, B_i, g_i), i \in I$, is an additive decomposition of $DP(A, B, g)$. Let $n_i = \dim(\mathcal{X}_i)$ and $m_i = \dim(\mathcal{E}_i)$. The complexity of solving $DP(A, B, g)$ is then $\sum_{i \in I} n_i \mathcal{E}_i = T q^{n+m}$.

**Proof**: Assume $(A, B, g)$ admits an additive decomposition. Then by Lemma 3 it is enough instead to solve $r$ smaller in general problems of dimensions $n_1, \ldots, n_r$ with $n_1 + \cdots + n_r = n, n_i = \dim(\mathcal{X}_i)$. Note also that for each problem the solution is sought over $\mathcal{E}_i$ and not the entire $\mathcal{U}$. Specifically, we only need to solve those problems for which the corresponding subspaces are affected by the input, i.e., $\mathcal{N}(B) \subseteq \mathcal{E}_i$, for if $\mathcal{N}(B) = \mathcal{E}_i$ for some $i \in I$, then the corresponding subsystem is unforced, thus we can set $u_i(x_i) = v_i, \forall t \in T, \forall x_i \in \mathcal{X}_i$, where $v_i$ is some arbitrarily selected element of $\mathcal{N}(B)$. Since the cardinality of $\mathcal{E}_i$ is $q^{m_i}$ for some $m_i \leq m$, the complexity of solving the $i^{th}$ problem, given that $\mathcal{E}_i \neq \mathcal{N}(B)$, is $T q^{n+m_i}$. 

**VI. Illustrative Examples**

**Example 1**: Denote by $Z_3$ the integers modulo 3 and consider the case where $\mathcal{X}$ is the 2-dimensional coordinate vector space over the field $Z_3$, i.e., $\mathcal{X} = (Z_3)^{2 \times 1}$ and $\mathcal{U} = Z_3$. Consider the system

$$x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t.$$  

The invariant subspaces of the state space are $\mathcal{X}_1 = < \begin{bmatrix} 1 \\ 0 \end{bmatrix} >_{Z_3}$, $\mathcal{X}_2 = < \begin{bmatrix} 1 \\ 1 \end{bmatrix} >_{Z_3}$ and we observe that $\mathcal{R}(B) = \mathcal{X}_2$, i.e. the condition of Thm. 1 is satisfied. Associate with this system some $g : Z_3^{2 \times 1} \to \mathbb{R}^+$ such that $g \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = g \begin{bmatrix} (\alpha_1 + 2\alpha_2) \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = g \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 0 \end{bmatrix} + g \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix},$ i.e. $g$ splits over the invariant subspaces. Since $\mathcal{E}_1 = \{0\}, \mathcal{E}_2 = Z_3$, the relevant subspaces are

$$x_{1,t+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_{1,t}$$  

$$x_{2,t+1} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} x_{2,t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_{2,t}$$  

with the second subsystem being isomorphic to the system

$$y_{t+1} = 2 y_t + \xi_t, \ y_t, \xi_t \in Z_3.$$  

Defining a cost function $h : Z_3 \to \mathbb{R}^+$ by

$$h(\alpha) = g \begin{bmatrix} \alpha \\ 0 \end{bmatrix},$$

we see that any optimal controller $\kappa^*_t$ for $(2,1,h)$ is also an optimal controller $u^*_t$ for
under the relation $u_t^\ast \left( \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \right) = \min_{u \in \mathbb{Z}_3} \left[ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right] \cdot \left[ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] + \min_{u \in \mathbb{Z}_3} (2\alpha_2 + u)\). 

As the above example suggests, the subspaces of the state-space that are relevant to the computation of an optimal controller, are only those that are affected by the input map, i.e. those for which $N(B) \subseteq \mathcal{E}_i$. 

**Example 2 (Linear Quadratic Control):** Consider the case where $X = \mathbb{R}^n, U = \mathbb{R}^m$, and the dynamics $x_{t+1} = Ax_t + Bu_t$ with $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$ over a finite horizon $t \in \{0, 1, \ldots, T\}, T \in \mathbb{Z}^+$. Let $g(x) = x^T P x$ where $P$ is positive-definite. Then $J_t^*(x) = x^T K_t x$ with $K_t = P$ and $K_i$ given by the backward algebraic Riccati recursion (assuming that $B$ has full column rank) 

$$K_i = P + A^T K_{t+1} A - A^T K_{t+1} B (B^T K_{t+1} B)^{-1} B^T K_{t+1} A$$ 

and the optimal controller is given by 

$$u_t^*(x) = -(B^T K_t B)^{-1} B^T K_i x.$$ 

Now let $A = SJS^{-1}$ be the canonical Jordan decomposition of $A$ and denote by $S_i$ the submatrix of $S$ consisting of the columns of $S$ corresponding to the $i^{th}$ Jordan block of $J$. Define $X_i = R(S_i)$. Then by construction $X_i$ is $A$-invariant and $\mathbb{R}^n = \bigoplus_{i \in \mathcal{I}} X_i$. Now suppose that the subspaces $X_i$ are $P$-orthogonal, i.e. $x_i^T P x_j = 0$, whenever $x_i \in X_i, x_j \in X_j, i \neq j$. This implies that $g \in G_s$. Additionally suppose that $R(B) = \bigoplus_{i \in \mathcal{I}} \{ R(B) \cap X_i \}$ and so $\mathbb{R}^m = \bigoplus_{i \in \mathcal{I}} \mathcal{E}_i$. Consider the representation of $A, P$ on a basis of $\mathbb{R}^n$ given by the union of basis of each of the subspaces $X_i$; then $P, A$ will be block diagonal. Moreover choosing as a basis of $\mathbb{R}^m$ the union of basis of the subspaces $\mathcal{E}_i$ and representing the image of $B$ using the above mentioned basis of $\mathbb{R}^m$, yields $B$ in a block diagonal form as well (even though $B$ need not be square). Then it is seen that the algebraic Riccati recursion becomes block diagonal, the block recursions representing the Riccati recursions corresponding to the subsystems $(A_i, B_i)$. Finally, the optimal controller itself is diagonal, each of its entries giving an optimal controller for the corresponding subsystem.

The following example shows that even though the structure of the principle of optimality is additive, a DP problem might admit a decomposition with the cost to go function decomposing not necessarily additively.

**Example 3:** As in example 1, let $X = (\mathbb{Z}_3)^{2 \times 1}$. Additionally, define $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in (\mathbb{Z}_3)^{2 \times 2}, \ i.e. X = X_1 \oplus X_2$ with $X_1 = \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \succ X_3, X_2 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \} \succ X_3$. 

Define $U = (\mathbb{Z}_3)^{2 \times 1}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in (\mathbb{Z}_3)^{2 \times 2}$ and a cost function $g : (\mathbb{Z}_3)^{2 \times 1} \to \mathbb{R}^+$ by $g(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = h(x_1) h(x_2)$ with $h : \mathbb{Z}_3 \to \mathbb{R}^+$ defined by $h(0) = 1, h(1) = 0, h(2) = 2$. Then $A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $g_1(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = g_2(\begin{bmatrix} 0 \\ x_2 \end{bmatrix}) = h(x_1) h(x_2)$. Hence $g(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = g_1(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) g_2(\begin{bmatrix} 0 \\ x_2 \end{bmatrix})$. Now, since $\min_{u \in \mathbb{Z}_3} g(A x + B u) = 0, \forall x \in X$, we have by induction that $J_t^* = g$ and similarly $J_{t+1}^* = g_1 J_{t+1}^* g_2$. Consequently $J_t^* = H(J_{t+1}^*, J_{t+1}^*) = J_{t+1}^* H_{t+1}^*$, and $\arg\min_{u \in \mathbb{Z}_3} J_{t+1}^* = \begin{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix}$ and $\arg\min_{u \in \mathbb{Z}_3} J_{t+1}^* = \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{bmatrix}$, i.e. the function $K$ is additive.

The final two examples show that the condition of Thm. 1 is only sufficient but not necessary; i.e. a decomposition might exist even if $R(B) \neq \bigoplus_{i \in \mathcal{I}} \{ R(B) \cap X_i \}$. 

**Example 4:** Let $X = (\mathbb{Z}_3)^{3 \times 1}, U = (\mathbb{Z}_3)^{2 \times 1}$ and consider the system 

$$x_{t+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_t.$$ 

The invariant subspaces are $X_1 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \succ \mathbb{Z}_3, X_2 = \{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \succ \mathbb{Z}_3, X_3 = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \succ \mathbb{Z}_3$. Note that $R(B) \cap X_1 = R(B) \cap X_3 = \{ 0 \}, R(B) \cap X_2 = X_2$. The three subsystems are 

$$(A_1, B_1) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$(A_2, B_2) = \begin{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$(A_3, B_3) = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}.$$ 

Since $(\mathbb{Z}_3)^{3 \times 1} = X_1 \oplus X_2 \oplus X_3$, every element of
can be written as a unique linear combination of
\[
\begin{bmatrix}
    1 & 0 & 0 \\
    1 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]
In particular, for any \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_3 \) we have that
\[
\begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \alpha_3
\end{bmatrix}
= (\alpha_1 + 2\alpha_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]
Now consider a cost function \( g : (\mathbb{Z}_3)^{3 \times 1} \to \mathbb{R}^+ \) with the property that \( g \left( \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \right) = g \left( \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right), \) i.e. \( g \) penalizes only \( X_2 \). Such a function satisfies the required splitting property. Consider also a finite horizon \( T = 1 \). Then the optimal controller corresponding to state \( [x_1 \ x_2 \ x_3] \) is given as the solution to the problem
\[
\min_{u_1, u_2 \in \mathbb{Z}_3} g \left( \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)
\]
which can equivalently be written as
\[
\min_{u_1, u_2 \in \mathbb{Z}_3} g \left( \begin{bmatrix}
    1 & 1 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ 2u_2 \\ u_2 \end{bmatrix} \right)
\]
or equivalently using the property of \( g \)
\[
\min_{u_1, u_2 \in \mathbb{Z}_3} g \left( \begin{bmatrix}
    1 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} + (u_1 + u_2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)
\]
which is equivalent to
\[
\min_{u_1 \in \mathbb{Z}_3} g \left( \begin{bmatrix}
    0 & 2 & 0 \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \right)
\]
the latter being precisely the problem giving the optimal controller of subsystem 2. Consequently, if \([u_1^* 0^*] \) is an optimal controller for subsystem 2 corresponding to state \([x_2 \ x_2 \ 0]^T \), then \([u_1^* 0^*] \) is also an optimal controller for the original system corresponding to state \([x_1 \ x_2 \ x_3]^T \) for any \( x_1, x_2, x_3 \in \mathbb{Z}_3 \).

Example 5: Similarly to example 4, consider the dynamics
\[
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 2 & 0 \\
    0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]
which is now \( \mathbb{R} \). Then the same argument as in example 4 can be applied, to show that if the cost function penalizes only the invariant subspace \( X_2 = \langle [1 \ 1 \ 0]^T \rangle \), the DP problem on the original system is equivalent to the DP problem on the subsystem corresponding to \( X_2 \).

VII. Future Work

Future work will be concerned with further refining the necessary and sufficient conditions, studying such conditions for decompositions more general than additive and considering the same questions for systems with different dynamics than the linear case that was considered in this paper.

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