On the Nature and Stability of Differential-Algebraic Systems

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Abstract

Nonlinear dynamical systems described by a class of higher index differential-algebraic equations (DAE) are considered. A quantitative and qualitative analysis of their nature and of the stability properties of their solution is presented. Using tools from geometric control theory, higher index differential-algebraic systems are shown to be inherently unstable about their solution manifold. A qualitative geometric interpretation is given, and the consequences of this instability are discussed, in particular as they relate to the numerical solution of these systems. The paper concludes with ideas and directions for future research.

1. Introduction

As the name indicates, differential-algebraic (DAE) systems are systems whose dynamical behavior is mathematically described by a combination of ordinary differential and algebraic equations:

\[ \dot{x} = f(x,z,t) \]  \hspace{1cm} (1.1)

\[ \theta = g(x,z,t) \]  \hspace{1cm} (1.2)

where the Jacobian of \( f \) with respect to \( x \) is nonsingular.

Interest in these systems first emerged in the early 1960's among researchers working on circuit models. Much research effort has since been directed towards studying these systems, both within the control and the numerical communities. The focus of the control community has been on extending the concepts and methods of modern control theory to this class of systems [4,7,12,20,21], while the focus of the numerical community has been on finding appropriate numerical schemes for simulating these systems [2]. Interestingly, the research within the two communities seems to have progressed along parallel tracks, with little cross exchange until recently.

Despite much progress made, a veil of ambiguity and confusion still seems to surround higher index DAE systems. In particular, the numerical community recognized early on that DAEs are fundamentally different from systems described by ordinary differential equations [15], and a quantitative measure of the singularity of the problem, the DAE differential index, was defined. However, the underlying causes and the nature of the difficulties haven't yet been clearly understood and explained. Hence, the quest for stable numerical schemes for simulating these systems has been plagued with difficulties [1]. The schemes that have been developed are only applicable for certain classes of systems [3], typically systems with index less than three. Moreover, many of these schemes suffer from degraded performance in the presence of even the slightest inconsistencies in initialization [14]. On the other hand, the control community has been successful in extending state-space concepts [19] and techniques [9,10,13] to LTI systems. Typically, the implied philosophy has been that these systems are more general forms of the standard state-space representation. Hence, they are seen to exhibit behaviors and properties common to state-space systems in addition to other interesting behaviors [18]. This is an obviously fruitful way of looking at these systems. However, an interesting question arises: Can DAE systems be alternatively considered as a special case of state-space systems, and if so, can any additional insight as to their true nature be gained by adopting this view? A related but more fundamental question is then: Are DAE systems an artifact due to the choice of representation or do they convey a more intrinsic property of the system which cannot be otherwise conveyed in the state-space representation?

This paper is meant to be a first step towards answering some of these questions and resolving some of this ambiguity through a mix of quantitative and qualitative analysis: The nature of the difficulties in dealing with these systems will be quantitatively examined, using the tools of geometric control theory. It will be shown, for a class of nonlinear problems, that systems represented by higher...
index DAEs are inherently unstable about their solution manifold. Hence, the DAE representation is definitely problematic for simulation purposes and potentially problematic for model based control purposes. Additionally, a qualitative geometric interpretation allowing further insight into these systems is presented, and a short discussion of the nature of DAE systems follows. The analysis and discussions presented pertain to continuous time systems that are "well-posed", as will be explained shortly.

2. An overview of DAEs

Consider a dynamical system described by the set of differential-algebraic equations (1.1) and (1.2), where \( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{z} \in \mathbb{R}^r \). \( \mathbf{f}(\mathbf{x}, \mathbf{z}, t) \) and \( \mathbf{g}(\mathbf{x}, \mathbf{z}, t) \) are vector valued functions of appropriate dimensions. The Jacobian of \( \mathbf{f} \) with respect to \( \mathbf{x} \) is assumed to be non-singular, while that of \( \mathbf{g} \) with respect to \( \mathbf{z} \) may or may not be singular.

The differential index of a DAE, if it exists, is defined as the minimum number of times that all or part of the algebraic constraints, equation (1.2), must be differentiated with respect to time in order to solve for \( \dot{\mathbf{x}} \) and \( \dot{\mathbf{z}} \) as a continuous function of \( \mathbf{x}, \mathbf{z} \) and \( t \) [2]. The definition is straightforward for the case where equation (1.2) is scalar. For the general case, a vector index can be defined for the system as follows:

\[
\nu_{v,i} = [\nu_1, \ldots, \nu_r]^T
\]

(2.1)

\( \nu_i \) refers to the minimum number of times the \( i \)th algebraic constraint equation needs to be differentiated. The differential index is then:

\[
\nu = \max_{i, j} \nu_{v,i}
\]

(2.2)

The differential index is a measure of the singularity of the system: a system whose index is larger than one is referred to as a higher index system and is typically known to be more problematic. Not all physically meaningful DAE systems have a well defined index. For instance, DAEs representing over or under determined kinematic systems do not have a well defined index. For LTI systems, if the index exists it is unique, and existence of a well-defined index implies that the equations admit a unique solution. For general nonlinear systems, the notion of index is local. In this paper, we restrict ourselves to the study of systems that we call "well-posed", i.e. those that have well defined and constant local indices throughout the space of interest.

DAEs arise in a wide variety of contexts, most notably: prescribed path constraint problems [8], singular perturbation problems, multiple body mechanics involving closed kinematic chains [6], discretization of PDEs [5] and modeling of large scale interconnected systems and networks [17].

3. The nature of the difficulty

The goal of this section is to clarify the nature of the difficulties encountered when working with differential-algebraic systems.

Consider a nonlinear dynamical system described by a class of differential-algebraic equations of the form:

\[
\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}, \mathbf{z}, t) + \mathbf{f}_2(\mathbf{x}, \mathbf{z})\mathbf{u}
\]

(3.1)

\[
0 = \mathbf{g}_1(\mathbf{x}, \mathbf{z}) + \mathbf{g}_2(\mathbf{x}, \mathbf{z})\mathbf{u}
\]

(3.2)

having output equations:

\[
\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{z})
\]

(3.3)

\( \mathbf{x} \in \mathbb{R}^n \), \( \mathbf{z} \in \mathbb{R}^r \), \( \mathbf{u} \in \mathbb{R}^r \) and \( \mathbf{y} \in \mathbb{R}^m \). \( \mathbf{f}_2 \) and \( \mathbf{g}_2 \) are appropriately sized matrix valued functions of \( \mathbf{x} \) and \( \mathbf{z} \), while \( \mathbf{f}_1, \mathbf{g}_1 \) and \( \mathbf{h} \) are appropriately sized vector valued functions of \( \mathbf{x} \) and \( \mathbf{z} \). All functions are assumed to be sufficiently differentiable.

In order to simplify the analysis, we restrict our attention to a subset of the above systems that satisfy the following two properties:

i) There is only one system variable. Hence equation (3.2) reduces to a scalar equation. Note that this simplifying assumption is made in the interest of transparency; dropping it should not add any fundamental difficulties to the analysis.

ii) The input \( \mathbf{u} \) does not enter into the algebraic constraint equation or the expression for its first \( \nu-1 \) time derivatives. The significance of this assumption will become clearer in Section 4 when a geometric interpretation of DAEs is discussed.

Proposition 1 Through a suitably defined local coordinate transformation, the DAE (3.1)-(3.3) with differential index \( \nu \) can always be reduced to the form:

\[
\dot{\mathbf{\Phi}} = \mathbf{J}_0 \mathbf{\Phi}
\]

(3.4)

\[
\dot{\mathbf{\Psi}} = \mathbf{F}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{u})
\]

(3.5)

\[
\mathbf{y} = \mathbf{H}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{u})
\]

(3.6)

where \( \mathbf{\Phi} \in \mathbb{R}^r \), \( \mathbf{\Psi} \in \mathbb{R}^{n+1} \), \( \mathbf{J}_0 \in \mathbb{R} \) is a nilpotent Jordan matrix.

Proof:
Denote the partial derivative of a scalar function, \( \lambda x,z \), with respect to \( x \) along an \( n \)-vector valued function \( f(x,z) \) as:

\[
L_t x,z = \frac{\partial}{\partial x} f(x,z)
\]  

(3.7)

It can be shown, by successive differentiation of the algebraic constraint equation and its first \(-1\) time derivatives, that assumption (ii) is mathematically described as follows:

Essentially, the new coordinates are defined to be the algebraic constraint equation and its first \(-1\) time derivatives. It is always possible to define an additional \( n+1 \)-coordinates:

\[
\Psi_i = fct_i x,z,u
\]  

(3.13)

\[
\Psi_{n+1} = fct_{n+1} x,z,u
\]  

such that the resulting mapping:

\[
\begin{bmatrix} x \\ z \end{bmatrix} \text{ mapping } \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}
\]  

(3.14)

is locally non-singular. Hence, this mapping constitutes an admissible local coordinate transformation. It follows from the way this coordinate transformation was defined that:

\[
\Phi_i = \Phi_i \text{ for } i = 1, \ldots, n
\]  

(3.15)

\[
\Phi = L_{t_1} g_1 x,z + L_{t_2} \Phi_1 g_1 x,z u + \frac{\partial L_{t_3} \Phi_1 g_1 x,z}{\partial z} z
\]  

(3.16)

Fortunately, we don't have to worry about the right hand side expression in equation (3.16). Assuming that the DAE model holds, we conclude that the trajectory followed by the system will always be consistent with the algebraic equation (3.2) and its relevant time derivatives, regardless of the input applied to the system. In other words, for every input applied, the solution of the DAE system (3.1)-(3.2) will be such that the right hand side of equation (3.16) is identically equal to 0. Hence, we have:

\[
\Phi = J \Phi
\]  

(3.4)

where \( J \) is a nilpotent Jordan matrix. Finally, the dynamic equations of the remaining coordinates and the output equations can be re-written in terms of the new coordinate system by substituting for \( x \Phi, \Psi, u \) and \( z \Phi, \Psi, u \) in the relevant equations.

Proposition 2 The set \( \Phi, \Psi \mid \Phi = 0 \) is invariant.

Proof: Obvious by direct substitution into equation (3.4).

Proposition 3 For the LTI case, the system described by equations (3.4)-(3.6) has at least repeated poles at the origin.

Proof: For the LTI case, the dynamic equations of the system in the transformed coordinates reduce to the general form:
\[
\begin{pmatrix}
\Phi \\
\dot{\Phi}
\end{pmatrix} =
\begin{bmatrix}
J_0 & 0 \\
A_1 & A_2
\end{bmatrix}
\begin{bmatrix}
\Phi \\
\dot{\Phi}
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
B
\end{bmatrix} u
\]  
(3.17)
\[
y = C_1 \Phi + C_2 + Du
\]  
(3.18)

The poles of the system are given by the eigenvalues of \( J_0 \) and \( A_2 \); hence there are at least \( r \) poles of the system at the origin, corresponding to the \( r \) eigenvalues of \( J_0 \) at 0.

**Proposition 4** The system described by equations (3.4)-(3.6) is unstable about its solution manifold for \( r \geq 2 \).

**Proof:**
First note that any solution of the DAE system (3.1)-(3.2) must satisfy the algebraic constraint, equation (3.2), for all times:

\[
g_1(x,z) = 0 \quad \forall t
\]  
(3.19)

Thus we conclude that:

\[
\frac{\partial^k g_1(x,z)}{\partial t^k} = 0 \quad \forall t, k = 1, \ldots, -1
\]  
(3.20)

Hence, any solution of the DAE system lies in the solution manifold \( S \):

\[
S \equiv \left\{ (x,z) \mid \frac{\partial^k g_1(x,z)}{\partial t^k} = 0, \quad k = 1, \ldots, -1 \right\}
\]  
(3.21)

Equivalently in the transformed coordinate system:

\[
S \equiv \Phi, \quad |\Phi = 0|
\]  
(3.22)

Notice that \( S \), the solution manifold, is identical to the invariant set considered in Proposition 2.

Now assume that at time \( t_p \), we perturb the solution of system (3.4)-(3.5) away from the solution manifold \( S \):

\[
\Phi(t_p) = \varepsilon_i, \quad i = 1, \ldots
\]  
(3.23)

where \( 0 < |\varepsilon_i| < 1 \).

If the system has index 1 \((i=1)\), we have:

\[
\phi_1 = 0
\]  
(3.24)

Hence:

\[
\phi_1 = \varepsilon \quad \forall t \geq t_p
\]  
(3.25)

In this case a small perturbation about the solution manifold does not cause the solution to move infinitely away from \( S \).

Thus we can conclude that this system is stable in the sense of Lyapunov about its solution manifold.

Now consider the higher index case \((i \geq 2)\). Again, perturb the system about its solution manifold \( S \) at time \( t_p \). We have:

\[
\phi_i = \varepsilon_i, \quad \forall t \geq t_p, \quad i = 1, \ldots, r
\]  
(3.26)

\[
\phi = 0 \quad \forall t \geq t_p
\]  
(3.27)

Hence for \( t \geq t_p \), \( \phi(t_p) = \varepsilon_i, \quad t_p \), \( i = 1, \ldots, r \), grows without bound, and the system moves infinitely away from its solution manifold, regardless of how we choose to reasonably define the distance between the solution trajectory and the solution manifold \( S \). Hence, the higher index system is unstable about its solution manifold, both in the sense of Lyapunov and in an asymptotic sense.

Obviously, Proposition 3 is a special instance of Proposition 4 where, for LTI systems, the source of instability about the solution manifold can be clearly traced to the existence of multiple poles at the origin.

### 4. Geometric Interpretation

Differential-algebraic equations are known to represent differential equations on a manifold [16]. Consider the DAE system (3.1)-(3.2) in vector space \( R^n \). The trajectory of the solution \( x^T, z^T \) of this system must satisfy the algebraic constraint equation (3.2) and its first \( 1 \) derivatives, as described in the proof of Proposition 4. Thus, the solution trajectory lies on the solution manifold, \( S \), defined in the previous section. In other words, the DAE equations are the representation of a dynamical system in a higher dimension state space \( x, z \).

Proposition 1 comes as no surprise then. The coordinate system transformation introduced clearly defines two orthogonal subspaces in the higher dimension space \( x, z \).

One subspace, spanned by \( \Phi' \), represents the directions perpendicular to the solution manifold at any operating point of the system. The other subspace, spanned by \( \Phi \), represents the direction tangent to the solution manifold at any operating point (Figure 1). Therefore, the variables can be interpreted as the "truly independent" variables, while the \( \Phi \) variables can be interpreted as the "redundant" variables. Essentially, this is saying that the system's dynamic behavior can be fully described by the
variables in the transformed coordinate system. Moreover, all we need in order to describe the system’s behavior in the original, redundant coordinate system is knowledge of 1) and the equation of the solution manifold $S$. This information would enable us to project the system trajectory along the $x,z$ coordinates.

![Coordinate transformation of Proposition 1](image)

Figure 1: Coordinate transformation of Proposition 1

Proposition 2 reflects the fact that $S$ is an invariant manifold in the sense that: for all initial conditions $\left( x_o^T, z_o^T \right)^T S$, the solution trajectory remains on the manifold $S$, regardless of the system input. Propositions 3 and 4 reflect the fact that the solution manifold $S$ is not attractive. Any perturbation of the solution away from $S$ will cause the solution trajectory to move infinitely away from the solution manifold (Figure 2).

![Perturbed solution trajectory](image)

Figure 2: Perturbed solution trajectory

Care should be taken not to incorrectly interpret these propositions. Perturbations of the solution of a mathematical model should not be confused with perturbations of the underlying physical system. In other words, if the DAE model of a system holds, we expect solutions corresponding to consistent initial conditions to remain on the solution manifold $S$, while we expect perturbed solutions to move away indefinitely. On the other hand, if the system itself is physically perturbed from its solution manifold, we have no clue what to expect as the DAE model itself may become invalid in the presence of such perturbations to the system.

Finally, it is to be cautioned that the issue of stability of the solution trajectory about an equilibrium point or trajectory on manifold $S$, which is indicative of the stability of the underlying physical system, is totally distinct from the issue of stability of the DAE representation about its solution manifold. While all higher index DAEs are unstable about $S$, the unperturbed solution trajectory starting on $S$ can end up at an equilibrium point or equilibrium trajectory of the system on $S$. Conversely, even though an index 1 DAE system is stable about $S$ in the sense of Lyapunov, this does not preclude the solution trajectory moving infinitely away from an equilibrium point on $S$ while remaining on the solution manifold.

5. Discussion

The stability properties of DAE systems as described in Section 3 are the source of trouble for the numerical community. Any inevitable numerical error or inconsistency in initialization results in the simulation blowing up unless something is done to stabilize it and bring it back to conform to the solution manifold. Hence the need for special numerical techniques and/or stable index reduction methods. The consequences of this instability when DAE models are used in a control setting are subtler. The principal issue there is the need and the ability (or inability) to differentiate between instability of the solution due to the choice of representation versus instability of the underlying physical system.

The geometric interpretation given in the previous section brings up some important issues to ponder. The DAE representation seems to be redundant, perhaps even a modeling artifact. Clearly that is the case for a simple pendulum: when polar coordinates are used, the pendulum equation is a second order ODE. When rectangular coordinates are used, the resulting equations are an index 3 DAE. Moreover, it seems that by appropriate choice of a coordinate system, the system dynamics can be completely described in a reduced order vector space tangent to the solution manifold at every point. If it is so wished, the solution trajectory can then be described in the original coordinate system by simple projection. The question is then: Is it always the case that the DAE representation is an artifact due to modeling assumptions? And if so, what are the conditions under which such a coordinate transformation is expected to exist? The answers to these questions are not known at this point, and the difficulty associated with obtaining such transformations is also not known in the general case. The next question is then: When
does it make sense to use the DAE representation in spite of
the associated “inconveniences”? It may be worth
investigating the DAE representation as a general
framework for modeling hybrid switching systems
where the system order is variable, as that would enable us
to represent and analyze a variable order system in one
redundant state space.

Finally, further work is needed in order to clarify: i) the
extent to which the propositions hold when the solution
manifold is dependent on the input and ii) the relationship,
if any, between the availability of local coordinate
transformations and the index, constant or changing of the
system.

6. Conclusions

A class of nonlinear differential-algebraic systems was
considered. For this class of systems, it was shown via an
appropriate coordinate transformation that the solution
of this representation is unstable about its solution manifold
when the system’s differential index is higher than 1. A
geometric interpretation was given for the coordinate
system transformation used and for the perceived
instability. It was noted that the coordinate transformation
used essentially decouples the state space into two
orthogonal subspaces: one tangent to the solution manifold
and another orthogonal to it. The corresponding coordinates
can be interpreted as the truly independent variables and the
redundant variables, respectively. Future research directions
were identified, namely: i) extending the results to more
general systems, ii) studying the conditions under which it
is possible to reduce a DAE to a minimal realization and iii)
the circumstances under which a DAE representation is
favorable, possibly as a general framework for hybrid
systems with variable order.

7. References

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